

## COMPLEX HÉNON MAPS AND DISCRETE GROUPS

RALUCA TANASE<sup>1</sup>

ABSTRACT. Consider the standard family of complex Hénon maps  $H(x, y) = (p(x) - ay, x)$ , where  $p$  is a quadratic polynomial and  $a$  is a complex parameter. Let  $U^+$  be the set of points that escape to infinity under forward iterations. The analytic structure of the escaping set  $U^+$  is well understood from previous work of J. Hubbard and R. Oberste-Vorth as a quotient of  $(\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}$  by a discrete group of automorphisms  $\Gamma$  isomorphic to  $\mathbb{Z}[1/2]/\mathbb{Z}$ . On the other hand, the boundary  $J^+$  of  $U^+$  is a complicated fractal object on which the Hénon map behaves chaotically. We show how to extend the group action to  $\mathbb{S}^1 \times \mathbb{C}$ , in order to represent the set  $J^+$  as a quotient of  $\mathbb{S}^1 \times \mathbb{C}/\Gamma$  by an equivalence relation. We analyze this extension for Hénon maps that are small perturbations of hyperbolic polynomials with connected Julia sets or polynomials with a parabolic fixed point.

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<sup>1</sup>INSTITUTE FOR MATHEMATICAL SCIENCES, STONY BROOK UNIVERSITY, STONY BROOK, NY 11794.

*E-mail address:* rtanase@math.sunysb.edu.

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## 1. INTRODUCTION

Hénon maps have played an important role in the development of modern dynamics, both in the real and in the complex setting. Real Hénon maps were first introduced by Michel Hénon as a simplified model of the Poincaré section of the Lorenz model. The dynamics of Hénon maps is intriguing and challenging and they are some of the most studied examples of dynamical systems that exhibit chaotic behavior. As a complex system, the Hénon map is also of major interest, due to the fact that all polynomial automorphisms of  $\mathbb{C}^2$  can be reduced to compositions of Hénon maps with simpler functions, as shown by S. Friedland and J. Milnor in [FM].

We consider the standard family of complex Hénon maps  $H_{p,a}(x, y) = (p(x) - ay, x)$ , where  $p$  is a quadratic polynomial and  $a$  is a complex parameter. Let  $U^+$  and  $U^-$  be the set of points that escape to infinity under forward and respectively backward iterations of the Hénon map. The topological boundaries  $J^+$  of  $U^+$  and  $J^-$  of  $U^-$  are complicated fractal sets on which the Hénon map behaves chaotically. The sets  $J^+$ ,  $J^-$  and  $J = J^+ \cap J^-$  are called the Julia sets of the Hénon map, and  $J$  is considered to be the analogue of the Julia set from one-dimensional dynamics.

This article is devoted to discrete group actions and connections with the topology of the set  $J^+$ . The analytic structure of the escaping set  $U^+$  is well understood from previous work of J. Hubbard and R. Oberste-Vorth in [HOV1] as a quotient of  $(\mathbb{C} - \mathbb{D}) \times \mathbb{C}$  by a discrete group of automorphisms  $\Gamma$  isomorphic to  $\mathbb{Z}[1/2]/\mathbb{Z}$ . As usual,  $\mathbb{D}$  denotes the open unit disk in the complex plane. We explain this result in Section 3.

In Section 4 we show how to extend the group action to the boundary  $\mathbb{S}^1 \times \mathbb{C}$  in certain cases, in order to represent the fractal set  $J^+$  as a quotient of  $\mathbb{S}^1 \times \mathbb{C}/\Gamma$  by an explicit equivalence relation. The group extension has important topological consequences that we describe in Section 9, where we analyze the extension for Hénon maps that are perturbations of hyperbolic polynomials with connected Julia set. In Theorem 9.3 we show that the group acts properly discontinuous and without fixed points on  $\mathbb{S}^1 \times \mathbb{C}$  and thus taking the quotient of  $\mathbb{S}^1 \times \mathbb{C}$  by the group action gives a topological manifold  $\mathcal{M}$ . The dynamics of the Hénon map on the set  $J^+$  is semi-conjugate to the dynamics of a model map on  $\mathcal{M}$ . The semi-conjugacy function can be viewed as a two-dimensional analogue of the Carathéodory loop from polynomial dynamics. In the simplest case, when  $p$  has an attractive fixed point ( $p$  is taken from the interior of the main cardioid of the Mandelbrot set), an actual conjugacy is achieved, so  $J^+$  itself is a topological manifold. In the other cases studied, we show that the set  $J^+$  is a quotient of the manifold  $\mathcal{M}$  by an equivalence relation which is described explicitly in Theorem 9.11.

The proof uses some results of M. Lyubich and J. Robertson [LR] on the characterization of the critical locus for complex Hénon maps. The proof also requires a careful analysis of the invariants of the Hénon map. In Section 4 we introduce an important function for the study of the Hénon family, which we denote  $\alpha : \mathbb{S}^1 \rightarrow \mathbb{C}^*$  and which encodes the dynamics of the Hénon map. The image of  $\alpha$  is a fractal set, for which we have designed and implemented a plotting algorithm in Section 7. In Section 6, we studied the degeneracy of the cocycle  $\alpha$  as the Jacobian tends to 0, and came up with an interesting relation connecting  $\alpha$  with the group action on  $\mathbb{S}^1 \times \mathbb{C}$ . Section 8 provides some sharp estimates of the growth of the group elements. These are useful for proving in Section 9 that the group acts properly discontinuously on  $\mathbb{S}^1 \times \mathbb{C}$ .

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## 2. PRELIMINARIES

Consider the complex Hénon map  $H_{p,a}(x, y) = (p(x) - ay, x)$ , where  $p(x) = x^2 + c$  is a monic polynomial of degree two. If  $a \neq 0$ ,  $H_{p,a}$  is a biholomorphism with constant Jacobian equal to  $a$ , and the inverse map is  $H_{p,a}^{-1}(x, y) = (y, (p(y) - x)/a)$ .

The *filled-in Julia set* of the polynomial  $p$  is defined as

$$K_p = \{z \in \mathbb{C} : |p^{\circ n}(z)| \text{ remains bounded as } n \rightarrow \infty\}.$$

The set  $J_p = \partial K_p$  is the *Julia set* of  $p$ . In analogy with one-dimensional dynamics, one defines the following dynamically invariant sets for the Hénon map:

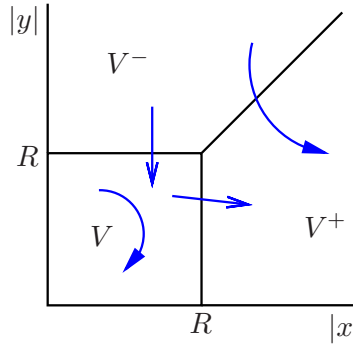
$$K^\pm = \{(x, y) \in \mathbb{C}^2 : \|H_{p,a}^{\circ n}(x, y)\| \text{ remains bounded as } n \rightarrow \pm\infty\}$$

$$U^\pm = \mathbb{C}^2 - K^\pm \quad (\text{the escaping sets})$$

$$J^\pm = \partial K^\pm = \partial U^\pm$$

$$K = K^- \cap K^+ \quad \text{and} \quad J = J^- \cap J^+.$$

The sets  $J^+$  and  $J^-$  are closed, unbounded, connected fractal objects in  $\mathbb{C}^2$  [BS1]. In the cases that we will be working with, the Jacobian  $a$  has absolute value less than 1, so  $K^-$  has no interior and  $J^- = K^-$ . When  $a$  is small and  $p(x) = x^2 + c$  is a hyperbolic polynomial, the interior of  $K^+$  consists of the basins of attraction of an attractive periodic orbit. The common boundary of the basins is  $J^+$  [FS], [BS1].



**Figure 1.** Filtration of  $\mathbb{C}^2$ .

According to [HOV1], for  $R > 2$  sufficiently large, the dynamical space  $\mathbb{C}^2$  can be divided into three regions:  $V = \{(x, y) \in \mathbb{C}^2 : |x| \leq R, |y| \leq R\}$ ,

$$V^+ = \{(x, y) : |x| \geq \max(|y|, R)\} \quad \text{and} \quad V^- = \{(x, y) : |y| \geq \max(|x|, R)\}.$$

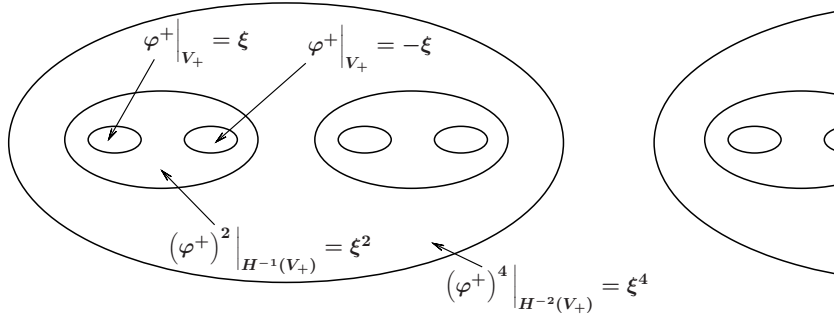
The sets  $J$  and  $K$  are contained in the polydisk  $V$ . The escaping sets can be described as union of backward iterates of  $V^+$  and respectively forward iterates of  $V^-$  under the Hénon map:

$$U^+ = \bigcup_{k \geq 0} H^{-\circ k}(V^+) \quad \text{and} \quad U^- = \bigcup_{k \geq 0} H^{\circ k}(V^-).$$

The domains  $V^+$  and  $V^-$  are easier to understand because one can define an analogue of the Böttcher coordinates. More precisely we have the following lemma:

**Lemma 2.1** (Hubbard, Oberste-Vorth [HOV1]). *There exists a unique holomorphic map  $\varphi^+ : V^+ \rightarrow \mathbb{C} - \overline{\mathbb{D}}$  such that  $\varphi^+ \circ H = (\varphi^+)^2$  and  $\varphi^+(x, y) \sim x$  as  $(x, y) \rightarrow \infty$  in  $V^+$ . There exists a unique holomorphic map  $\varphi^- : V^- \rightarrow \mathbb{C} - \overline{\mathbb{D}}$  such that  $\varphi^- \circ H^{-1} = (\varphi^-)^2$  and  $\varphi^-(x, y) \sim y$  as  $(x, y) \rightarrow \infty$  in  $V^-$ .*

The set  $U^+$  is foliated by copies of  $\mathbb{C}$ , which have a natural affine structure. The holomorphic function  $\varphi^+$  defines a holomorphic foliation on  $V^+$ . The leaves of the foliation are just the level sets of  $\varphi^+$ . One can then extend this foliation from  $V^+$  to  $U^+$  by the dynamics. The function  $(\varphi^+)^{2^k}$  is well defined on  $H^{-\circ k}(V^+)$  as  $(\varphi^+)^{2^k} = \varphi^+ \circ H^{\circ k}$  and it defines a holomorphic foliation on  $H^{-\circ k}(V^+)$ .



**Figure 2.** A fiber  $\mathcal{F}_\xi$  of the foliation of  $U^+$ , for  $\xi \in \mathbb{C} - \overline{\mathbb{D}}$ .

One can also define a similar holomorphic foliation of the set  $U^-$  using the map  $\varphi^-$ . The foliations of the escaping sets  $U^+$  and  $U^-$  are not everywhere transverse to each other. The critical locus  $\mathcal{C}$  of the Hénon map is the set of tangencies between the foliation of  $U^+$  and the foliation of  $U^-$ . The set  $\mathcal{C}$  is a closed analytic subvariety of  $U^+ \cap U^-$  and is invariant under the Hénon map.

**Theorem 2.2** (Bedford, Smillie [BS5]). *The critical locus  $\mathcal{C}$  is nonempty. The boundary  $\partial\mathcal{C}$  of  $\mathcal{C}$  intersects both  $J^+$  and  $J^-$  and we have  $\overline{\mathcal{C}} \cap J^+ \cap U^- \neq \emptyset$  and  $\overline{\mathcal{C}} \cap J^- \cap U^+ \neq \emptyset$ .*

**Theorem 2.3** (Lyubich, Robertson [LR]). *Let  $H$  be a hyperbolic Hénon map with connected  $J$ , which is a small perturbation of a hyperbolic polynomial  $p(x) = x^2 + c$ , with connected Julia set  $J_p$ . We have the following description of the critical locus:*

- (a) *There exists a unique primary component  $\mathcal{C}_0$  of the critical locus asymptotic to the  $x$ -axis.*
- (b) *There exists a biholomorphic extension of  $\varphi^+$  from  $\mathcal{C}_0$  to  $\mathbb{C} - \overline{\mathbb{D}}$ .*
- (c) *Moreover, there exists a biholomorphism  $\tau^+$  from  $\mathcal{C}_0$  to  $\mathbb{C} - K_p$ , which can be extended homeomorphically from  $\overline{\mathcal{C}}_0$  to  $\mathbb{C} - \overset{\circ}{K}_p$ .*
- (d)  *$\mathcal{C}_0$  is everywhere transverse to the foliation of  $U^+$  and  $U^-$ .*
- (e) *All other components of  $\mathcal{C}$  are forward or backward iterates of  $\mathcal{C}_0$  under  $H$ .*

**Remark 2.4.** Since the Hénon map is hyperbolic with connected Julia set, the boundary of  $\mathcal{C}_0$  belongs to  $J^+$ . The forward iterates of  $\mathcal{C}_0$  accumulate on  $J^-$ .

**Remark 2.5.** A model for the critical locus is also described in [F] for perturbations of quadratic hyperbolic polynomials with disconnected Julia sets. The critical locus is connected in this case.

**The degenerate case  $a = 0$ .** The picture when  $a$  is 0 helps visualize the foliation of  $U^+$  and  $J^+$  and the primary component of the critical locus. The Hénon map  $H_{p,0}(x, y) = (p(x), x)$  is no longer a biholomorphism and maps all  $\mathbb{C}^2$  to the curve  $\{x = p(y)\}$ . However, the foliations of  $U^+$  and  $J^+$  persist and are easier to describe:

- (a)  $\varphi^+$  is just the Böttcher isomorphism of  $p$ .
- (b)  $J^+ = J_p \times \mathbb{C}$ , where  $J_p$  is the Julia set of  $p$ .
- (c)  $U^+ = (\mathbb{C} - K_p) \times \mathbb{C}$ , where  $K_p$  is the filled-in Julia set of  $p$ .

The primary component of the critical locus can also be easily understood from [LR],  $\bar{\mathcal{C}}_0 = (\mathbb{C} - \mathring{K}_p) \times \mathbb{C}$ , where  $\mathring{K}_p$  is the interior of the filled-in Julia set of  $p$ .

### 3. THE COVERING SPACE OF THE ESCAPING SET $U^+$

In this section we describe the analytic structure of the escaping set  $U^+$ .

**Lemma 3.1.** *There exists a closed holomorphic 1-form on  $U^+$ , with  $H^*w = 2w$ .*

**Proof.** The map  $\varphi^+$  is well defined on  $U^+$  up to local choices of roots of unity, so  $\log(\varphi^+)$  is well defined up to local addition of constants. Hence the form  $w = d \log \varphi^+$  is well defined and holomorphic on  $U^+$ . It is easy to see from Lemma 2.1 that

$$H^*w = H^* \frac{d\varphi^+}{\varphi^+} = \frac{d(\varphi^+ \circ H)}{\varphi^+ \circ H} = \frac{d((\varphi^+)^2)}{\varphi^+} = 2w. \quad \square$$

**Definition 3.2.** For a closed curve  $C$  in  $U^+$ , define the index  $\eta(C)$  as

$$\eta(C) := \frac{1}{2\pi i} \int_C w. \quad (1)$$

Since  $w$  is a closed 1-form, the number  $\eta(C)$  depends only on the homotopy type of  $C$ . The following properties from [BS8] and [MNTU] of  $\eta$  are helpful for understanding the topology of the escaping set  $U^+$ :

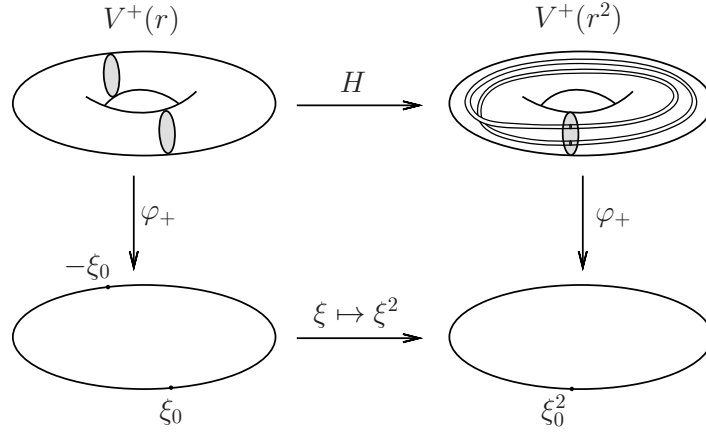
- (a)  $\eta(H^k(C)) = \frac{1}{2\pi i} \int_{H^k(C)} w = \frac{1}{2\pi i} \int_C (H^k)^*w = 2^k \eta(C)$ .
- (b) Take  $C \subset V^+$ . One can homotopically enlarge  $C$  so that it belongs to the region where  $\varphi^+(x, y) \sim x$ . Then  $\eta(C) = \frac{1}{2\pi i} \int_C \frac{dx}{x} \in \mathbb{Z}$ , since it represents the winding number of  $C$  around the  $x$ -axis.
- (c) In  $U^+$ , there exists  $k$  such that  $H^k(C) \subset V^+$ , so  $\eta(H^k(C)) = m \in \mathbb{Z}$ . Therefore  $\eta(C) = \frac{m}{2^k}$ , where  $m$  and  $k$  are integer numbers.

**Lemma 3.3** ([HOV1]). *The fundamental group of  $U^+$  is isomorphic to  $\mathbb{Z}[\frac{1}{2}]$  where  $\mathbb{Z}[\frac{1}{2}] = \{\frac{m}{2^k} \mid m, k \in \mathbb{Z}\}$ .*

**Proof.** The proof is an immediate consequence of the properties (a), (b) and (c) listed above.  $\square$

We would like of course to be able to extend  $\varphi^+$ , and not only  $d \log \varphi^+$  to the whole set  $U^+$ . However there are topological obstructions which become clear once we look at the behavior of the Hénon map near infinity in  $V^+$ .

**Lemma 3.4** ([HOV1]). *For large  $r$ , the set  $V^+(r) = \{(x, y) \in V^+, |\varphi^+| = r\}$  is homeomorphic to a solid torus, and  $\varphi^+ : V^+(r) \rightarrow \{z, |z| = r\}$  is a fibration with fibers homeomorphic to closed disks. On  $V^+$  the Hénon map is solenoidal, and the following diagram commutes:*



The map  $\varphi^+$  does not extend holomorphically to  $U^+$ , but it does extend along curves contained in  $U^+$  which start in  $V^+$  so it is well defined on a covering manifold  $\tilde{U}^+$  of  $U^+$ . The covering manifold  $\tilde{U}^+$  is called the Riemann surface of  $\varphi^+$  and its construction is fairly standard, nonetheless, for completion, we will outline an explicit construction of  $\tilde{U}^+$  from [MNTU].

One starts by fixing a base point  $a \in V^+$ , and defines the set  $\tilde{U}^+$  as follows:

$$\begin{aligned} \tilde{U}^+ &= \{(z, C), z \in U^+, C \text{ is a path in } U^+ \text{ between } a \text{ and } z\} / \sim \\ &\text{where } (z, C) \sim (z', C') \Leftrightarrow z = z' \text{ and } \eta(CC'^{-1}) \in \mathbb{Z}. \end{aligned}$$

The definition does not depend on the choice of a particular representative  $(z, C)$  for the equivalence class. If  $(z', C')$  is another representative, then  $z = z'$  and  $\eta(CC'^{-1}) = m \in \mathbb{Z}$ . It follows that  $\int_C w - \int_{C'} w = 2\pi i m$ , so  $\eta([z, C]) = \eta([z', C'])$ . There is also an analogue of  $V^+$  in  $\tilde{V}^+$ , represented by the set

$$\tilde{V}^+ = \{[z, C] \in \tilde{U}^+, z \in V^+, C \subset V^+\}.$$

One can then define a lift of  $\varphi^+$  to the covering manifold  $\tilde{U}^+$ . Let  $\tilde{\varphi}^+ : \tilde{U}^+ \rightarrow \mathbb{C} - \mathbb{D}$  be given by the relation

$$\tilde{\varphi}^+([z, C]) = \varphi^+(a) \exp \int_C w.$$

It is easy to show that

$$\tilde{\varphi}^+|_{\tilde{V}^+} = \varphi^+|_{V^+} \quad \text{and} \quad \tilde{U}^+ = \bigcup_{n \geq 0} \tilde{H}^{-\circ n}(\tilde{V}^+). \quad (2)$$

To check the first equality from Relation 2, take an equivalence class  $[z, C] \in \tilde{V}^+$  such that  $z \in V^+$ . One can verify by direct computation that

$$\tilde{\varphi}^+([z, C]) = \varphi^+(a) e^{\log(\varphi^+(z)) - \log(\varphi^+(a))} = \varphi^+(z).$$

To show the second part of Relation 2, take an equivalence class  $[z, C] \in \tilde{H}^{-\circ n}(\tilde{V}^+)$  such that  $z \in H^{-\circ n}(V^+)$ . Then we have

$$\tilde{\varphi}^+([z, C]) = \varphi^+(a) \exp \int_C w = \varphi^+(a) \exp \left( \frac{1}{2^n} \int_{H^{\circ n} C} w \right)$$

and it follows that

$$\begin{aligned} \tilde{\varphi}^+([z, C])^{2^n} &= \varphi^+(a)^{2^n} e^{\log(\varphi^+(H^{\circ n}(z))) - \log(\varphi^+(H^{\circ n}(a)))} \\ &= \varphi^+(H^{\circ n}(z)) = \varphi^+(z)^{2^n}. \end{aligned} \quad (3)$$

**Theorem 3.5** (Hubbard, Oberste-Vorth [HOV1]). *The covering manifold  $\tilde{U}^+$  is a trivial analytic fiber bundle over  $\mathbb{C} - \mathbb{D}$ , with fibers isomorphic to  $\mathbb{C}$ .*

A nice proof of this theorem is given in [HOV1] and we will not reproduce it here in detail. The key point of the proof is to show that the map  $\tilde{\varphi}^+ : \tilde{U}^+ \rightarrow (\mathbb{C} - \mathbb{D})$  is an analytic submersion with fibers isomorphic to  $\mathbb{C}$ , then show (by a nontrivial argument) that  $\tilde{U}^+$  is a locally trivial fiber bundle, locally homeomorphic to  $(\mathbb{C} - \mathbb{D}) \times \mathbb{C}$ . Then the result of Theorem 3.5 follows by complex analysis, as  $\mathbb{C} - \mathbb{D}$  is Stein, so topological and analytic classification of line bundles over  $\mathbb{C} - \mathbb{D}$  coincide.

$$\begin{array}{ccc} \tilde{U}^+ & & \\ \pi \downarrow & \searrow \tilde{\varphi}^+ & \\ U^+ \supset V^+ & \xrightarrow{\varphi^+} & \mathbb{C} - \mathbb{D} \end{array}$$

**Theorem 3.6** (Hubbard, Oberste-Vorth [HOV1]). *The analytic structure of  $U^+$  is well-understood:*

$$U^+ = (\mathbb{C} - \mathbb{D}) \times \mathbb{C} / \Gamma_{p,a},$$

where  $\Gamma_{p,a} \subset \text{Aut}((\mathbb{C} - \mathbb{D}) \times \mathbb{C})$  is a discrete group isomorphic to  $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ .

**Proof.** By Theorem 3.5,  $\tilde{U}^+$  is a covering manifold of  $U^+$ , hence one can describe  $U^+$  as a quotient of  $\tilde{U}^+$  by a group  $\Gamma_{p,a}$  of deck transformations. The group  $\Gamma_{p,a}$  is isomorphic

to  $\pi_1(U^+)/\pi_1(\tilde{U}^+)$ , hence isomorphic to  $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$  by Lemma 3.3. The following diagram depicts the situation:

$$\begin{array}{ccc} \tilde{U}^+ & \xrightarrow{\simeq} & (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C} \\ \pi \downarrow & & \downarrow \pi \\ U^+ & \xrightarrow{\simeq} & (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}/\Gamma_{p,a} \end{array} \quad \square$$

There is a unique lift  $\tilde{H}$  of  $H$  to the covering manifold  $\tilde{U}^+$  such that the following diagram commutes and conditions 1-4 hold:

$$\begin{array}{ccc} \tilde{U}^+ & \xrightarrow{\tilde{H}} & \tilde{U}^+ \\ \pi \downarrow & & \downarrow \pi \\ U^+ & \xrightarrow{H} & U^+ \end{array} \quad \begin{array}{l} 1. \pi \circ \tilde{H} = H \circ \pi \\ 2. \tilde{\varphi}^+ \circ \tilde{H} = (\tilde{\varphi}^+)^2 \text{ on } \tilde{U}^+ \\ 3. \tilde{H} \circ \gamma = (2\gamma) \circ \tilde{H}, \text{ for all } \gamma \in \Gamma_{p,a} \\ 4. \tilde{H}(\tilde{V}^+) \subset \tilde{V}^+ \end{array}$$

The map  $\tilde{H}$  is a covering map from  $\tilde{U}^+$  to  $\tilde{U}^+$  with sheet number 2.

**Remark 3.7.** The foliation of the covering manifold  $\tilde{U}$  by level sets of the function  $\tilde{\varphi}^+$  descends to a foliation of the escaping set  $U^+$ . This is the same foliation as the one induced by the function  $\varphi^+$  on  $U^+$  in Section 2, as it can be easily seen from relations 2 and 3 and the properties of the lift  $\tilde{H}$ .

#### 4. THE STABLE MULTIPLIER CONDITION

We will show how to extend the group action  $\Gamma_{p,a}$  to  $\mathbb{S}^1 \times \mathbb{C}$  in certain cases, in order to represent the fractal boundary  $J^+$  of  $U^+$  as a quotient of  $\mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a}$  by an equivalence relation. We will discuss this in Theorem 9.4 and Corollary 9.5.1.

Let us first explain the meaning of an extension of the group elements to  $\mathbb{S}^1 \times \mathbb{C}$ . After a particular trivialization of the covering manifold  $\tilde{U}^+$  has been chosen, one can define a lift of the Hénon map to  $(\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}$  so that the following diagram commutes

$$\begin{array}{ccc} (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C} & \xrightarrow{\tilde{H}} & (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C} \\ \pi \downarrow & & \downarrow \pi \\ U^+ & \xrightarrow{H} & U^+ \end{array}$$

It follows from conditions 1-3 that the lift  $\tilde{H}$  of the Hénon map is an analytic function of the form

$$\tilde{H}(\xi, z) = (\xi^2, \alpha(\xi)z + \beta(\xi)), \quad (4)$$

where  $\alpha : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C}^*$  and  $\beta : \mathbb{C} - \overline{\mathbb{D}} \rightarrow \mathbb{C}$  are analytic functions.

**Extension.** We would like to extend the map  $\tilde{H}$  to  $\mathbb{S}^1 \times \mathbb{C}$ , so that the dynamics of  $\tilde{H}$  on  $\mathbb{S}^1 \times \mathbb{C}$  is "compatible" with the dynamics of  $H$  on the Julia set  $J^+$ . However, the set  $J^+$  contains stable manifolds of periodic points in  $J$  (as we will see in Theorem 4.2), so we must require that the following condition is satisfied:



**Stable Multiplier Condition.** The functions  $\alpha$  and  $\beta$  extend continuously to  $\mathbb{S}^1$ . The stable multipliers of  $\tilde{H}$  on  $\mathbb{S}^1 \times \mathbb{C}$  agree with the stable multipliers of  $H$  on  $J^+$ , in the sense that for every periodic point  $\xi = \xi^{2^k}$  of the doubling map on  $\mathbb{S}^1$  there exists a  $k$ -periodic point  $x$  of the Hénon map  $H$  such that

$$\alpha(\xi)\alpha(\xi^2)\dots\alpha(\xi^{2^{k-1}}) = \lambda(DH_x^{\circ k}), \quad (5)$$

where  $\lambda$  is the small eigenvalue of  $DH_x^{\circ k}$  at  $x$ .

We will call a function  $\alpha$  which satisfies the Stable Multiplier Condition a cocycle.

**Remark 4.1.** In [HOV1], in addition to the description of the covering manifold  $\tilde{U}^+$ , it is also shown that there exists a unique isomorphism  $\tilde{U}^+ \rightarrow (\mathbb{C} - \mathbb{D}) \times \mathbb{C}$  such that, with this trivialization, the map  $\tilde{H}$  is written as

$$\tilde{H}(\xi, z) = \left( \xi^2, \frac{a}{2}z + \xi^3 - \frac{c}{2}\xi \right).$$

Notice that even if there is no problem in continuously extending this map to  $\mathbb{S}^1 \times \mathbb{C}$ , the dynamics of the extension  $\tilde{H}$  on  $\mathbb{S}^1 \times \mathbb{C}$  is quite different from the dynamics of the Hénon map  $H$  on  $J^+$ . The stable multipliers of  $\tilde{H}$  are “too simple”, as they are all equal to  $a/2$ , whereas the multipliers of the Hénon map can (and will) be complicated.

In Section 2, we described the foliation of the escaping set  $U^+$ . When the Hénon map  $H$  is hyperbolic, the boundary  $J^+$  of  $U^+$  is also foliated by copies of  $\mathbb{C}$ , given by stable manifolds of points from the Julia set  $J$ , as illustrated by the following theorem:

**Theorem 4.2** (Bedford, Smillie [BS7]).

- (a) Let  $p \in J$  be a saddle periodic point of the Hénon map, then  $J^+$  is the closure of  $W^s(p)$ .
- (b) For hyperbolic Hénon maps with Jacobian  $|a| < 1$ , the set  $J^+ = W^s(J)$ , so  $J^+$  has its own dynamically defined Riemann surface lamination, whose leaves consist of the stable manifolds  $W^s(p)$  of points  $p \in J$ .
- (c) If in addition, the Julia set  $J$  is connected, then the foliation of  $U^+$  and the lamination of  $J^+$  fit together continuously to form a locally trivial lamination of  $U^+ \cup J^+$ .

When  $H$  is hyperbolic, for each point  $p \in J$  there exists a biholomorphic function  $\varphi : \mathbb{C} \rightarrow W^s(p)$  which defines an affine structure on the stable manifold  $W^s(p)$ . In addition, the iterates of the Hénon map  $H$  preserve the affine structure, in the sense that the pull-back or push-forward of the affine structure from one leaf to another agrees with the original affine structure on the new leaf [BS5].

## 5. A TRIVIALIZATION OF THE LAMINATION OF $U^+ \cup J^+$

We will use the primary component of the critical locus from Theorem 2.3 to give an identification of each of the fibers  $\mathcal{F}_\xi = (\tilde{\varphi}^+)^{-1}(\xi)$  with  $\mathbb{C}$ . These identifications will provide a specific trivialization of the bundle  $\tilde{U}^+ \simeq (\mathbb{C} - \mathbb{D}) \times \mathbb{C}$ .

It follows from Theorem 2.3 that the primary component  $\mathcal{C}_0$  of the critical locus is biholomorphic to the exterior of the filled-in Julia set  $K_p$  of the polynomial  $p$  via a map  $\tau^+ : \mathcal{C}_0 \rightarrow \mathbb{C} - K_p$  that extends to a homeomorphism between the boundary of  $\mathcal{C}_0$  and

the Julia set  $J_p$ . Therefore, the closure of the primary component  $\bar{\mathcal{C}}_0$  can be naturally identified with  $\mathbb{C} - \mathbb{D}$  via the map

$$\mathbb{C} - \mathbb{D} \xrightarrow{\gamma} \mathbb{C} - \overset{\circ}{K}_p \xrightarrow{(\tau^+)^{-1}} \bar{\mathcal{C}}_0,$$

where the composition  $(\tau^+|_{\bar{\mathcal{C}}_0})^{-1} \circ \gamma$  is biholomorphic on  $\mathbb{C} - \bar{\mathbb{D}}$  and continuous on  $\mathbb{C} - \mathbb{D}$ . The function  $\gamma$  in the diagram above is the Böttcher coordinate of the polynomial  $p$ .

We briefly recall the definition of the Böttcher coordinate from [DH] and [M]. Let  $p$  be a quadratic polynomial with connected filled-in Julia set  $K_p$ . There exists a unique analytic map  $\varphi : \mathbb{C} - K_p \rightarrow \mathbb{C} - \bar{\mathbb{D}}$  tangent to the identity at infinity that conjugates  $p$  to  $z \rightarrow z^2$ , that is  $\varphi \circ p = (\varphi)^2$ . The function  $\varphi$  is called the *Böttcher isomorphism*, and the inverse map  $\gamma = \varphi^{-1} : \mathbb{C} - \bar{\mathbb{D}} \rightarrow \mathbb{C} - K_p$  the *Böttcher coordinate*. If in addition the filled-in Julia set  $K_p$  is locally connected, the Böttcher coordinate extends continuously to  $\gamma : \mathbb{S}^1 \rightarrow J_p$ ,  $t \rightarrow \lim_{r \rightarrow 1^+} \gamma(re^{2\pi it})$ . The extension is a continuous surjective map called the *Carathéodory loop*.

**Lemma 5.1 (Trivialization lemma).** *There exists a continuous surjective function  $\pi : (\mathbb{C} - \mathbb{D}) \times \mathbb{C} \rightarrow U^+ \cup J^+$ , holomorphic from  $(\mathbb{C} - \bar{\mathbb{D}}) \times \mathbb{C} \rightarrow U^+$  and analytic on the leaves of the lamination of  $J^+$ , such that the following diagram commutes*

$$\begin{array}{ccc} (\mathbb{C} - \mathbb{D}) \times \mathbb{C} & \xrightarrow{\tilde{H}} & (\mathbb{C} - \mathbb{D}) \times \mathbb{C} \\ \pi \downarrow & & \downarrow \pi \\ U^+ \cup J^+ & \xrightarrow{H} & U^+ \cup J^+ \end{array}$$

where  $\tilde{H}(\xi, z) = (\xi^2, \alpha(\xi)z + \beta(\xi))$ , and the functions  $\alpha : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}^*$  and  $\beta : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}$  are continuous on  $\mathbb{C} - \mathbb{D}$  and analytic on  $\mathbb{C} - \bar{\mathbb{D}}$ .

**Proof.** As in [BV], we will construct holomorphic parametrizations of the leaves of the foliation of  $U^+$  that converge locally uniformly to the parametrization of a limit leaf of the lamination of  $J^+$ . Let  $\mathcal{F}_\xi$  be a leaf of the lamination of  $U^+ \cup J^+$ . The critical points  $c_0(\xi)$  and  $c_{-1}(\xi)$  belong to  $\mathcal{F}_\xi$  and they are given by the relation

$$c_0(\xi) = (\tau^+|_{\bar{\mathcal{C}}_0})^{-1} \circ \gamma(\xi) \quad \text{and} \quad c_{-1}(\xi) = H^{-1}(c_0(\xi^2)). \quad (6)$$

Each leaf is biholomorphic to  $\mathbb{C}$  and there exists a unique analytic mapping  $\pi_\xi : \mathbb{C} \rightarrow \mathcal{F}_\xi$  which sends

$$0 \rightarrow c_0(\xi) \quad \text{and} \quad 1 \rightarrow c_{-1}(\xi). \quad (7)$$

We can therefore define the function  $\pi : (\mathbb{C} - \mathbb{D}) \times \mathbb{C} \rightarrow U^+ \cup J^+$  by  $\pi(\xi, z) = \pi_\xi(z)$ .

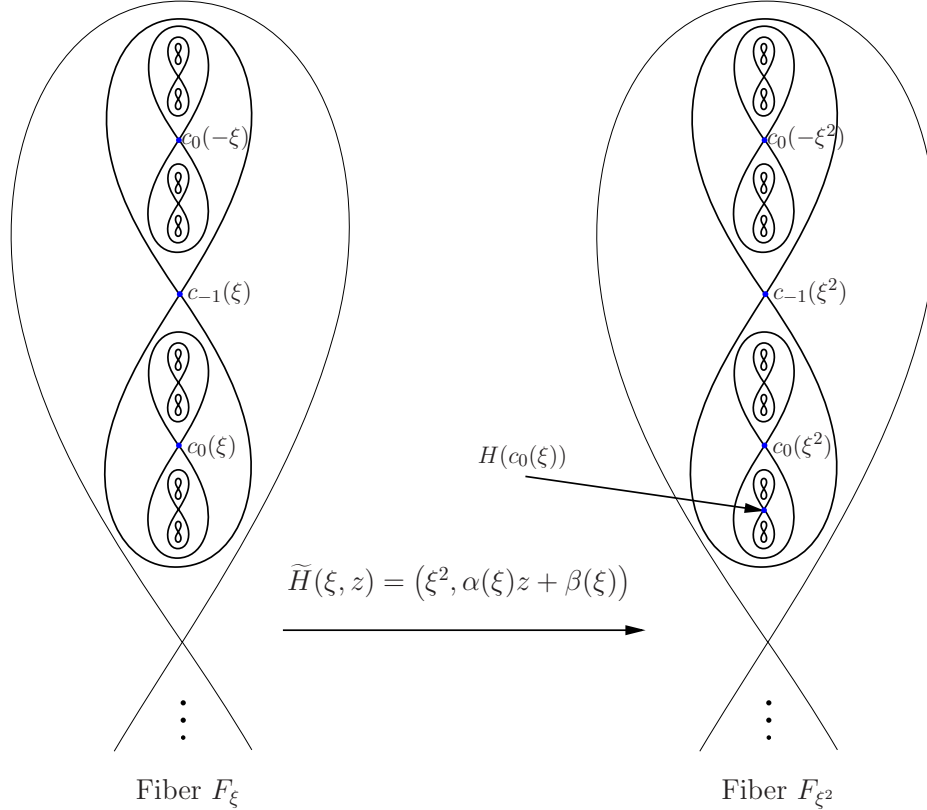
Recall from Section 2 that  $\mathcal{F}_\xi = \mathcal{F}_{\omega\xi}$  for all  $\omega$  dyadic roots of unity ( $\omega$  is dyadic if  $\omega^{2^k} = 1$ , for some non-negative integer  $k$ ). Of course, the primary component  $\bar{\mathcal{C}}_0$  of the critical locus intersects  $\mathcal{F}_\xi$  at all points of the form  $c_0(\omega\xi)$ , where  $\omega^{2^k} = 1$  for some integer  $k \geq 0$ . Therefore we will end up parametrizing the same fiber  $\mathcal{F}_\xi$  "a dyadic number of times". We parametrize  $\mathcal{F}_{\omega\xi}$  by first fixing  $c_0(\omega\xi)$  at the origin. The nearest intersection point of  $\bar{\mathcal{C}}_{-1}$  with  $\mathcal{F}_\xi$  will then be  $c_{-1}(\omega\xi)$ , and we set this to be 1.

The function  $c_0 : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}^2$  is holomorphic on  $\mathbb{C} - \overline{\mathbb{D}}$  and continuous on  $\mathbb{S}^1$ , as shown in Theorem 2.3. Consequently  $c_{-1} : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}^2$  has the same properties. The primary component of the critical locus  $\overline{\mathcal{C}}_0$  is transverse to the leaves on the foliation of  $U^+$ , by Theorem 2.3. The affine structure on  $J^+ \cup U^+$  is transversely continuous [BS5]. Hence the projection  $\pi$  is continuous on  $(\mathbb{C} - \mathbb{D}) \times \mathbb{C}$ .

The fact that the function  $\pi$  is analytic on  $(\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}$  follows from the construction of the covering manifold  $\widetilde{U}^+$ . It is worth noting at this stage that  $\pi : (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C} \rightarrow U^+$  is a covering map, but  $\pi : \mathbb{S}^1 \times \mathbb{C} \rightarrow J^+$  is not in general a covering map, unless the Hénon map is a perturbation of a quadratic polynomial with an attractive fixed point.

The Hénon map becomes  $\widetilde{H}(\xi, z) = (\xi^2, \alpha(\xi)z + \beta(\xi))$ . For a fixed  $\xi$ , we compute  $\alpha(\xi)$  and  $\beta(\xi)$  by looking at the affine structures on the fibers  $\mathcal{F}_\xi$  and  $\mathcal{F}_{\xi^2}$ .

$$\begin{aligned} \alpha(\xi)[c_{-1}(\xi)]_{\mathcal{F}_\xi} + \beta(\xi) &= [c_0(\xi^2)]_{\mathcal{F}_{\xi^2}} \\ \alpha(\xi)[c_0(\xi)]_{\mathcal{F}_\xi} + \beta(\xi) &= [H(c_0(\xi))]_{\mathcal{F}_{\xi^2}} \end{aligned} \tag{8}$$



**Figure 3.** Two fibers  $\mathcal{F}_\xi$  and  $\mathcal{F}_{\xi^2}$  of the lamination of  $J^+ \cup U^+$  and the action of the map  $\widetilde{H}$  on the critical points  $c_0(\xi)$  and  $c_{-1}(\xi)$ .

The fiber  $\mathcal{F}_{\xi^2}$  is biholomorphic to  $\mathbb{C}$ , hence the ratio of the points  $c_0(\xi^2)$ ,  $H(c_0(\xi))$  and  $c_{-1}(\xi^2)$  does not depend on the choice of affine maps on  $\mathcal{F}_{\xi^2}$ . Hence if we denote by

$[H(c_0(\xi))]_{\mathcal{F}_{\xi^2}}$  the coordinate of the point  $H(c_0(\xi))$  with respect to the particular affine map on  $\mathcal{F}_{\xi^2}$  which assigns  $c_0(\xi^2) \rightarrow 0$  and  $c_{-1}(\xi^2) \rightarrow 1$ , we obtain

$$[H(c_0(\xi))]_{\mathcal{F}_{\xi^2}} = -\frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - c_{-1}(\xi^2)}$$

and we can compute

$$\alpha(\xi) = -\beta(\xi) = \frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - H^{-1}(c_0(\xi^4))}. \quad (9)$$

Clearly  $\alpha$  does not vanish. Otherwise  $\bar{\mathcal{C}}_0$  and  $\bar{\mathcal{C}}_1$  would intersect and this is not possible from [LR]. More precisely, the primary component  $\bar{\mathcal{C}}_0$  is inside a trapping region around the  $x$ -axis that contains no other components of the critical locus.  $\square$

**Remark 5.2.** We would like to write  $c_1(\xi^2)$  in place of  $H(c_0(\xi))$  in Equation 8, but this would be incorrect, as we would not be able to distinguish between  $H(c_0(\xi))$  and  $H(c_0(-\xi))$ , which are two distinct points of  $\mathcal{F}_{\xi^2}$  (see also Figure 3).

**Remark 5.3.** In the Trivialization Lemma 5.1, we could have worked with the  $x$ -axis in place of the primary component of the critical locus. However, when  $|a|$  is big, there is no reason to assume that the  $x$ -axis will remain transverse to the foliation of  $U^+$ . Choosing a transverse which has dynamical meaning,  $\mathcal{C}_0$ , gives hope of extending the results to the whole interior of the hyperbolic component of the Hénon connectedness locus that contains perturbations of a hyperbolic polynomial. In fact, Theorem 2.3 is also believed to hold in this generality.

**Proposition 5.4.** *The function  $\alpha : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}^*$  is well defined, analytic on  $\mathbb{C} - \bar{\mathbb{D}}$  and continuous on  $\mathbb{S}^1$ .*

**Proof.** For  $\xi$  fixed,  $\alpha(\xi)$  is defined in Equation 9 as the difference quotient of three points  $H(c_0(\xi))$ ,  $c_0(\xi^2)$  and  $c_{-1}(\xi^2)$  from  $\mathcal{F}_{\xi^2}$ . The ratio  $(x - y)/(y - z)$  of three distinct points  $x, y, z$  from a manifold biholomorphic to  $\mathbb{C}$  is independent of the choice of a particular trivialization. Hence  $\alpha$  is well defined. The function  $c_0 : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}^2$  is holomorphic on  $\mathbb{C} - \bar{\mathbb{D}}$  and continuous on  $\mathbb{S}^1$ . The affine structure on  $J^+ \cup U^+$  is transversely continuous [BS5].  $\square$

**Proposition 5.5.** *The function  $\alpha$  is unique up to multiplication by appropriate maps of the form  $u(\xi^2)/u(\xi)$ , where  $u : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}^*$  is holomorphic on  $\mathbb{C} - \bar{\mathbb{D}}$  and continuous on  $\mathbb{S}^1$ . In addition, the function  $u(\xi)$  is well defined on the primary component of the critical locus, that is, if  $c_0(\xi_1) = c_0(\xi_2)$  then  $u(\xi_1) = u(\xi_2)$ .*

**Proof.** Suppose we define another trivialization of  $\mathcal{F}_{\xi}$  that assigns  $c_0(\xi) \rightarrow v(\xi)$  and  $c_{-1}(\xi) \rightarrow u(\xi)$ , where  $u, v : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}$  and  $u(\xi) \neq v(\xi)$ . We use two very special transverses to give a trivialization of  $J^+ \cup U^+$ , namely  $\mathcal{C}_0$  and  $H^{-1}(\mathcal{C}_0)$ , which carry their own identifications, described in Theorem 2.3. So the assignments  $c_0(\xi) \rightarrow v(\xi)$  and  $c_{-1}(\xi) \rightarrow u(\xi)$  must preserve these identifications, that is, if  $c_0(\xi_1) = c_0(\xi_2)$  for some  $\xi_1, \xi_2 \in \mathbb{S}^1$  then  $u(\xi_1) = u(\xi_2)$  and  $v(\xi_1) = v(\xi_2)$ .

The same computation as before yields

$$\alpha(\xi)u(\xi) + \beta(\xi) = v(\xi^2) \quad \text{and} \quad \alpha(\xi)v(\xi) + \beta(\xi) = x,$$

where  $x$  is the coordinate of  $H(c_0(\xi))$  in  $\mathcal{F}_{\xi^2}$  and can be computed from the invariance property of the ratio of three points under affine changes of coordinates. We have

$$\frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - c_{-1}(\xi^2)} = \frac{x - v(\xi^2)}{v(\xi^2) - u(\xi^2)} \Rightarrow x = \frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - c_{-1}(\xi^2)} (v(\xi^2) - u(\xi^2)) + v(\xi^2).$$

After solving the system we get

$$\alpha(\xi) = \frac{v(\xi^2) - x}{u(\xi) - v(\xi)} = \frac{u(\xi^2) - v(\xi^2)}{u(\xi) - v(\xi)} \cdot \frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - c_{-1}(\xi^2)} \quad (10)$$

$$\beta(\xi) = v(\xi^2) - u(\xi) \cdot \frac{u(\xi^2) - v(\xi^2)}{u(\xi) - v(\xi)} \cdot \frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - c_{-1}(\xi^2)} \quad (11)$$

so the expression of  $\alpha$  has only changed by a multiplicative factor of the form

$$\frac{u(\xi^2) - v(\xi^2)}{u(\xi) - v(\xi)}. \quad (12)$$

Since we are looking only for functions  $\alpha(\xi)$  which are analytic on  $\mathbb{C} - \overline{\mathbb{D}}$  and continuous on  $\mathbb{S}^1$ , the maps  $u(\xi)$  and  $v(\xi)$  must also be holomorphic on  $\mathbb{C} - \overline{\mathbb{D}}$  and continuous on  $\mathbb{S}^1$ . We denote the multiplicative factor 12 by  $u(\xi^2)/u(\xi)$  when there is no danger of confusion.  $\square$

We will now show that the Stable Multiplier Condition 5 is satisfied.

**Proposition 5.6.** *For  $\xi \in \mathbb{S}^1$ ,  $\xi = \xi^{2^k}$ , the product  $\alpha(\xi)\alpha(\xi^2)\dots\alpha(\xi^{2^{k-1}})$  does not depend on the choices of affine maps and it equals the small eigenvalue  $\lambda(DH_x^{\circ k})$  of the derivative of  $H^{\circ k}$  at some  $k$ -periodic point  $x$  of  $H$ .*

**Proof.** Let  $\xi \in \mathbb{S}^1$ ,  $\xi = \xi^{2^k}$  be a periodic point of the doubling map  $\xi \rightarrow \xi^2$ .

The fiber  $\mathcal{F}_\xi$  is invariant under the Hénon map since  $H^{\circ k}(\mathcal{F}_\xi) = \mathcal{F}_{\xi^{2^k}} = \mathcal{F}_\xi$ , hence  $\mathcal{F}_\xi$  is the stable manifold  $W^s(x)$  of some periodic point  $x$  of period  $k$  of the Hénon map, and  $\mathcal{F}_{\xi^{2^i}}$  is the stable manifold of  $H^{\circ i}(x)$ ,  $1 \leq i \leq k$ . Moreover, since  $H$  is a hyperbolic Hénon map, we know that the tangent space  $T_{H^{\circ i}(x)}\mathcal{F}_{\xi^{2^i}}$  is the eigenspace of the smallest eigenvalue of the Jacobian matrix  $DH_{H^{\circ i}(x)}^{\circ i}$ .

The function  $\alpha(\xi)$  is unique up to a multiplicative factor  $u(\xi^2)/u(\xi)$ . We notice that

$$\frac{u(\xi^2)}{u(\xi)} \frac{u(\xi^4)}{u(\xi^2)} \dots \frac{u(\xi^{2^k})}{u(\xi^{2^{k-1}})} = 1,$$

hence the product  $\alpha(\xi)\alpha(\xi^2)\dots\alpha(\xi^{2^{k-1}})$  is well defined and independent of choices of affine maps on the fibers  $\mathcal{F}_\xi, \mathcal{F}_{\xi^2}, \dots, \mathcal{F}_{\xi^{2^{k-1}}}$ . Therefore

$$\alpha(\xi)\alpha(\xi^2)\dots\alpha(\xi^{2^{k-1}}) = \prod_{i=1}^k \frac{H(c_0(\xi^{2^{i-1}})) - c_0(\xi^{2^i})}{c_0(\xi^{2^i}) - H^{-1}(c_0(\xi^{2^{i+1}}))},$$

where each of the ratios is evaluated in  $\mathcal{F}_{\xi^{2^i}}$ ,  $1 \leq i \leq k$ .

Each fiber is biholomorphic to  $\mathbb{C}$  and we can choose convenient parametrizing functions

$$\psi_i : \mathbb{C} \rightarrow \mathcal{F}_{\xi^{2^i}} \quad \text{with} \quad \psi_i(0) = H^{\circ i}(x) \quad \text{and} \quad \psi'_i(0) = v_i,$$

where  $v_0$  is a stable eigenvector of  $DH_x$  and  $v_i = DH_{H^{\circ i}(x)}^{\circ i} v$ . Denote by  $\phi_i : \mathcal{F}_{\xi^{2^i}} \rightarrow \mathbb{C}$  the inverse function of  $\psi_i$ .

$$\begin{array}{ccccccc} \mathcal{F}_\xi & \xrightarrow{H} & \mathcal{F}_{\xi^2} & & \mathcal{F}_{\xi_{2^{k-1}}} & \xrightarrow{H} & \mathcal{F}_{\xi^{2^k}} \\ \phi_0 \downarrow & & \downarrow \phi_1 & \cdots & \downarrow \phi_{k-1} & & \downarrow \phi_k = \phi_0 \\ \mathbb{C} & \xrightarrow{L_1(z)=m_1 \cdot z} & \mathbb{C} & & \mathbb{C} & \xrightarrow{L_k(z)=m_k \cdot z} & \mathbb{C} \end{array}$$

The Hénon map induces multiplicative maps between the copies of  $\mathbb{C}$ ,  $\phi_i \circ H = m_i \cdot \phi_{i-1}$  where  $m_i \neq 0$  and we have

$$\phi_0 \circ H^{\circ k} = (m_k \cdot m_{k-1} \cdot \dots \cdot m_1) \cdot \phi_0.$$

If we differentiate the previous relation and evaluate at  $x$  we get

$$\nabla \phi_0 \cdot DH_x^{\circ k} \cdot v_0 = (m_k \cdot m_{k-1} \cdot \dots \cdot m_1) \cdot \nabla \phi_0 \cdot v_0.$$

But  $DH_x^{\circ k} \cdot v_0 = \lambda v_0$ , where  $\lambda$  with  $|\lambda| < 1$  is the small eigenvalue of the Jacobian matrix  $DH_x^{\circ k}$ . Hence

$$m_k \cdot m_{k-1} \cdot \dots \cdot m_1 = \lambda.$$

One can now compute the product

$$\begin{aligned} \prod_{i=1}^k \frac{\phi_i \circ H(c_0(\xi^{2^{i-1}})) - \phi_i \circ c_0(\xi^{2^i})}{\phi_i \circ c_0(\xi^{2^i}) - \phi_i \circ H^{-1}(c_0(\xi^{2^{i+1}}))} &= \prod_{i=1}^k \frac{\phi_i \circ H(c_0(\xi^{2^{i-1}})) - \phi_i \circ H \circ H^{-1}(c_0(\xi^{2^i}))}{\phi_i \circ c_0(\xi^{2^i}) - \phi_i \circ H^{-1}(c_0(\xi^{2^{i+1}}))} \\ &= \prod_{i=1}^k \frac{m_i \cdot \phi_{i-1} \circ c_0(\xi^{2^{i-1}}) - m_i \cdot \phi_{i-1} \circ H^{-1}(c_0(\xi^{2^i}))}{\phi_i \circ c_0(\xi^{2^i}) - \phi_i \circ H^{-1}(c_0(\xi^{2^{i+1}}))} \\ &= (m_1 \cdot m_2 \cdot \dots \cdot m_k) \frac{\phi_0 \circ c_0(\xi) - \phi_0 \circ H^{-1}(c_0(\xi^2))}{\phi_k \circ c_0(\xi^{2^k}) - \phi_k \circ H^{-1}(c_0(\xi^{2^{k+1}}))} = \lambda. \end{aligned} \quad \square$$

The description of candidate functions  $\alpha(\xi)$  from proposition 5.5 that satisfy the condition in 5.6 can be linked to other results of this sort.

**Theorem 5.7** (Livschitz [K]). *If  $\Lambda$  is a topologically transitive hyperbolic set for a diffeomorphism  $f$  and  $\varphi : \Lambda \rightarrow \mathbb{R}$  is a  $\tau$ -Hölder continuous function such that*

$$\sum_{i=0}^{n-1} \varphi(f^i(x)) = 0 \quad \text{whenever} \quad f^n(x) = x,$$

*then  $\varphi$  is a coboundary, i.e. there exists a continuous function  $h : \Lambda \rightarrow \mathbb{R}$  such that  $\varphi = h \circ f - h$ . This function is unique up to an additive constant, and it is a  $\tau$ -Hölder continuous.*

**Proposition 5.8.** *Suppose  $\xi_1, \xi_2$  are two points on  $\mathbb{S}^1$  such that  $\gamma(\xi_1) = \gamma(\xi_2)$ , where  $\gamma$  is the Carathéodory loop of the polynomial  $p$ . Then  $\alpha(\xi_1) = \alpha(\xi_2)$ .*

**Proof.** We first show that

$$\frac{H(c_0(\xi_1)) - c_0(\xi_1^2)}{c_0(\xi_1^2) - H^{-1}(c_0(\xi_1^4))} = \frac{H(c_0(\xi_2)) - c_0(\xi_2^2)}{c_0(\xi_2^2) - H^{-1}(c_0(\xi_2^4))}. \quad (13)$$

Since  $c_0(\xi) = \left(\tau^+|_{\bar{\mathbb{C}}_0}\right)^{-1} \circ \gamma(\xi)$  and  $\gamma(\xi_1) = \gamma(\xi_2)$  we have  $c_0(\xi_1) = c_0(\xi_2)$ . By using the properties of the Böttcher coordinate  $\gamma(\xi^2) = p(\gamma(\xi))$ ,  $\xi \in \mathbb{C} - \mathbb{D}$ , we get that  $c_0(\xi_1^2) = c_0(\xi_2^2)$  and  $c_0(\xi_1^4) = c_0(\xi_2^4)$ . Hence the two ratios in Equation 13 are equal.

By equation 10 in Proposition 5.5, the choice of  $\alpha$  is unique up to multiplication by functions of the form  $u(\xi^2)/u(\xi)$ . These functions  $u$  satisfy the additional property that  $c_0(\xi_1) = c_0(\xi_2) \Rightarrow u(\xi_1) = u(\xi_2)$ . By Theorem 2.3,  $c_0(\xi_1) = c_0(\xi_2)$  if and only if  $\gamma(\xi_1) = \gamma(\xi_2)$ .  $\square$

**Proposition 5.9.**  $\int_{\mathbb{S}^1} \log |\alpha(e^{2\pi i \theta})| d\theta = \lambda^-(\mu)$ , where  $\lambda^-(\mu)$  is the stable Lyapunov exponent with respect to the unique measure  $\mu$  of maximal entropy supported on the Julia set  $J$ .

**Proof.** The stable and unstable Lyapunov exponent are well understood in the case of hyperbolic Hénon maps. They are related by the equation

$$\lambda^+(\mu) + \lambda^-(\mu) = \log |DH| = \log |a|.$$

When  $|a| < 1$  and the Hénon map is hyperbolic with connected Julia set, the unstable Lyapunov exponent is  $\lambda^+(\mu) = \log(2)$ , as shown in [BS5]. Hence  $\lambda^-(\mu) = \log |a| - \log 2$ . The stable Lyapunov exponent  $\lambda^-$  is defined as

$$\lambda^- = \lim_{k \rightarrow \infty} \frac{1}{k} \log \|DH^{\circ k}|_{E^s}\|.$$

By [BS5], for  $\mu$  almost every point  $x$  in  $J$ , one has

$$\lambda^- = \lim_{k \rightarrow \infty} \frac{1}{k} \log \|DH_x^{\circ k}|_{E_x^s}\|.$$

Let  $\mathcal{F}_\xi$  be a leaf of the lamination of  $J^+$  that contains  $x$ . We can compute  $\lambda^-$  as follows

$$\begin{aligned} \lambda^- &= \lim_{k \rightarrow \infty} \frac{1}{k} \log \|DH_x^{\circ k}|_{E_x^s}\| = \lim_{k \rightarrow \infty} \frac{1}{k} \log |\alpha(\xi)\alpha(\xi^2) \cdots \alpha(\xi^{2^{k-1}})| \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \left( \log |\alpha(\xi)| + \log |\alpha(\xi^2)| + \cdots + \log |\alpha(\xi^{2^{k-1}})| \right) = \int_{\mathbb{S}^1} \log |\alpha(\xi)| d\theta. \end{aligned}$$

Here  $\int_{\mathbb{S}^1} \log |\alpha(\xi)| d\theta$  stands for  $\int_{\mathbb{S}^1} \log |\alpha(e^{2\pi i \theta})| d\theta$ , where  $d\theta$  is the Lebesgue measure on the unit circle  $\mathbb{S}^1$  regarded here as  $\mathbb{R}/\mathbb{Z}$ . The doubling map  $f(\xi) = \xi^2$  is ergodic with respect to the Lebesgue measure on  $\mathbb{S}^1$ , so the orbit of almost every  $\xi$  is equidistributed on  $\mathbb{S}^1$ . The last equality then follows from the Birkhoff Ergodic Theorem.  $\square$

**Remark 5.10.** Notice that  $\int_{\mathbb{S}^1} \log |\alpha(\xi)| d\theta$  in Lemma 5.9 does not depend on the choices involved in the construction of the function  $\alpha$ . The map  $\alpha$  is unique up to multiplication by a factor of the form  $u(\xi^2)/u(\xi)$ , where the map  $u : \mathbb{S}^1 \rightarrow \mathbb{C}^*$  is continuous. Since  $f(\xi) = \xi^2$  is measure preserving with respect to the Lebesgue measure on  $\mathbb{S}^1$ , we have  $\int_{\mathbb{S}^1} \log |u(\xi^2)| d\theta = \int_{\mathbb{S}^1} \log |u(\xi)| d\theta$ .

The function  $\alpha$  is probably a full invariant of the (quadratic) Hénon map, in the sense that if two hyperbolic Hénon maps  $H_1$  and  $H_2$  have the property that  $\alpha_1 = \alpha_2$  then the Hénon maps coincide, i.e.  $H_1 = H_2$ . The following proposition from [T] provides support for this claim.

**Proposition 5.11.** *The values  $\alpha(1)$  and  $\alpha(e^{2\pi i/3}) \cdot \alpha(e^{2\pi i 2/3})$  determine the Hénon map up to three choices.*

Moreover, by Proposition 5.9 we have  $\log |a| - \log(2) = \int_{\mathbb{S}^1} \log |\alpha(e^{2\pi i \theta})| d\theta$ , therefore the absolute value of the Jacobian is determined by the function  $\alpha$ . It would also be interesting to study the relation between the function  $\alpha$  and the non-transversality locus invariant (ntl-invariant) described in [HOV3].

## 6. DEGENERACY OF THE FUNCTION $\alpha$

It is easy to see that the limit of the function  $\alpha$  is zero when the Jacobian goes to zero. This is a consequence of the fact that the critical points on the primary component  $\mathcal{C}_0$  and on  $\mathcal{C}_1 = H(\mathcal{C}_0)$  remain bounded as  $a \rightarrow 0$  and close to the  $x$ -axis, respectively to the parabola  $\{(x, y) \in \mathbb{C}^2, x = p(y)\}$ . Meanwhile, by the definition of  $H^{-1}$  from Section 2, the critical points on  $\mathcal{C}_{-1} = H^{-1}(\mathcal{C}_0)$  go to infinity as the Jacobian tends to 0.

It is therefore more interesting and useful to compute the limit of  $\alpha(\xi)/a$  as  $a \rightarrow 0$ . In the trivialization that assigns  $c_0(\xi) \rightarrow 0$  and  $c_{-1}(\xi) \rightarrow 1$  we have computed in Equation 9 the following formula

$$\alpha(\xi) = \frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - H^{-1}(c_0(\xi^4))}. \quad (14)$$

Throughout this section, we will refer to trivialization 7 as the *standard trivialization* and to the function  $\alpha$  from Equation 14 as the function  $\alpha$  computed with respect to the standard trivialization.

**Proposition 6.1 (Degeneracy of  $\alpha$ ).**

$$\lim_{a \rightarrow 0} \frac{1}{a} \cdot \frac{H(c_0(\xi)) - c_0(\xi^2)}{c_0(\xi^2) - H^{-1}(c_0(\xi^4))} = \frac{\gamma(\xi)}{2(\gamma(\xi^2))^2}$$

where  $\gamma$  is the Böttcher coordinate of the polynomial  $p$ .

**Proof.** Let  $\xi \in \mathbb{C} - \mathbb{D}$  and set  $x = c_0(\xi)$ . The leaf  $\mathcal{F}_\xi$  is isomorphic to  $\mathbb{C}$  and one can choose a biholomorphic map

$$\psi_x : \mathbb{C} \rightarrow \mathcal{F}_\xi$$

such that  $\psi_x(0) = x$  and  $\psi'_x(0) = v$ , where  $v \in T_x \mathcal{F}_\xi$  is the unit vector  $v$ , with  $\|v\| = 1$  and  $pr_2(v) \in \mathbb{R}^+$ .

We know that when  $a$  is small,  $\mathcal{F}_\xi$  is almost vertical in a neighborhood of  $x$  [HOV2] [T]. Hence there exists a unique analytic function  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that locally around the critical point  $x = (f(z_0), z_0)$ , the leaf  $\mathcal{F}_\xi$  is the graph of  $f$ , of the form  $\{(f(z), z)\}$ . So we can choose  $v = (f'(z_0), 1)/\|(f'(z_0), 1)\|$ .

The leaves  $\mathcal{F}_{\xi^2}$  and  $\mathcal{F}_{\xi^2^2}$  are isomorphic to  $\mathbb{C}$ , hence there exist biholomorphisms

$$\psi_i : \mathbb{C} \rightarrow \mathcal{F}_{\xi^{2^i}}, \quad i = 1, 2$$



such that  $\psi_i(0) = H^{\circ i}(x)$  and  $\psi'_i(0) = w_i$ , where  $w_i \in T_{H^{\circ i}(x)}\mathcal{F}_{\xi^{2^i}}$  is a tangent vector with norm  $\|w_i\| = 1$  and  $pr_2(w_i) \in \mathbb{R}^+$ .

$$\begin{array}{ccccc} \mathcal{F}_\xi & \xrightarrow{H} & \mathcal{F}_{\xi^2} & \xrightarrow{H} & \mathcal{F}_{\xi^4} \\ \psi_x \uparrow & & \uparrow \psi_1 & & \uparrow \psi_2 \\ \mathbb{C} & \xrightarrow{L_1(z)=m_1 \cdot z} & \mathbb{C} & \xrightarrow{L_2(z)=m_2 \cdot z} & \mathbb{C} \end{array}$$

From this commutative diagram we get

$$\begin{aligned} H \circ \psi_1(z) &= \psi_2(m_2 \cdot z) \\ \psi_1^{-1} \circ H^{-1} &= \frac{1}{m_2} \cdot \psi_2^{-1}. \end{aligned}$$

We can therefore compute the function  $\alpha(\xi)$  using Equation 14 and get

$$\alpha(\xi) = \frac{\psi_1^{-1} \circ H(c_0(\xi)) - \psi_1^{-1} \circ c_0(\xi^2)}{\psi_1^{-1} \circ c_0(\xi^2) - \psi_1^{-1} \circ H^{-1}(c_0(\xi^4))} = \frac{\psi_1^{-1} \circ H(c_0(\xi)) - \psi_1^{-1} \circ c_0(\xi^2)}{\psi_2^{-1} \circ H(c_0(\xi^2)) - \psi_2^{-1} \circ c_0(\xi^4)} \cdot m_2.$$

When the Jacobian  $a$  is 0, the primary component of the critical locus degenerates uniformly to  $(\mathbb{C} - \mathring{K}_p) \times \{0\}$ , where  $K_p$  is the filled-in Julia set of the polynomial  $p$ . Moreover  $c_0(\xi) = (\gamma(\xi), 0)$ ,  $c_0(\xi^2) = (\gamma(\xi^2), 0)$  and  $c_0(\xi^4) = (\gamma(\xi^4), 0)$ . With our notation  $x = c_0(\xi)$ , we can also compute the degeneracy of the points  $H(x)$  and  $H^{\circ 2}(x)$ :

$$\begin{aligned} H(x) &= (\gamma(\xi)^2 + c, \gamma(\xi)) = (\gamma(\xi^2), \gamma(\xi)) \\ H^{\circ 2}(x) &= (\gamma(\xi^2)^2 + c, \gamma(\xi^2)) = (\gamma(\xi^4), \gamma(\xi^2)). \end{aligned}$$

The leaf  $\mathcal{F}_{\xi^2}$  degenerates to a collection of vertical lines

$$\left\{ c_0(\omega \xi^2) \times \mathbb{C} : \omega^{2^i} = 1, \text{ for some integer } i \geq 1 \right\}.$$

However, the parametrizing function  $\psi_1$  degenerates to the parametrization of the vertical line that passes through  $c_0(\xi^2)$ ,

$$\psi_1 : \mathbb{C} \rightarrow \gamma(\xi) \times \mathbb{C}, \quad \psi_1(z) = (\gamma(\xi^2), z + \gamma(\xi)).$$

The parametrizing function  $\psi_2$  degenerates to

$$\psi_2 : \mathbb{C} \rightarrow \gamma(\xi^2) \times \mathbb{C}, \quad \psi_2(z) = (\gamma(\xi^4), z + \gamma(\xi^2)).$$

Hence

$$\lim_{a \rightarrow 0} \frac{\psi_1^{-1} \circ H(c_0(\xi)) - \psi_1^{-1} \circ c_0(\xi^2)}{\psi_2^{-1} \circ H(c_0(\xi^2)) - \psi_2^{-1} \circ c_0(\xi^4)} = \frac{\gamma(\xi)}{\gamma(\xi^2)}. \quad (15)$$

From the commutative diagram we also know that

$$DH_{\psi_1(z)} \cdot \psi'_1(z) = m_2 \cdot \psi'_2(m_2 \cdot z).$$

When  $z = 0$  we have  $DH_{H(x)} \cdot \psi'_1(0) = m_2 \cdot \psi'_2(0)$ , or equivalently

$$\psi'_1(0) = m_2 \cdot DH_{H(x)}^{-1} \cdot \psi'_2(0). \quad (16)$$

The Hénon map is  $H(x_1, x_2) = (x_1^2 + c - ax_2, x_1)$  and the inverse has the formula  $H^{-1}(x_1, x_2) = (x_2, (x_2^2 + c - x_1)/a)$ , so

$$DH_x^{-1} = \begin{pmatrix} 0 & 1 \\ -1/a & 2x_2/a \end{pmatrix}.$$

It follows that

$$DH_{H(x)}^{-1} = \begin{pmatrix} 0 & 1 \\ -1/a & 2(x_1^2 + c - ax_2)/a \end{pmatrix}$$

and Equation 16 becomes

$$\psi'_1(0) = \frac{m_2}{a} \cdot \begin{pmatrix} 0 & a \\ -1 & 2(x_1^2 + c - ax_2) \end{pmatrix} \cdot \psi'_2(0). \quad (17)$$

Notice also that  $\lim_{a \rightarrow 0} \psi'_1(0) = \lim_{a \rightarrow 0} \psi'_2(0) = (0, 1)$ . Thus from Equation 17 we get

$$\lim_{a \rightarrow 0} \frac{m_2}{a} = \frac{1}{2\gamma(\xi^2)}. \quad (18)$$

Therefore from the relations 18 and 15 we can conclude that

$$\lim_{a \rightarrow 0} \frac{\alpha(\xi)}{a} = \frac{\gamma(\xi)}{\gamma(\xi^2)} \cdot \frac{1}{2\gamma(\xi^2)} = \frac{\gamma(\xi)}{2(\gamma(\xi^2))^2}.$$

An important observation is that the convergence is uniform in  $\xi$ . This follows as a consequence of the fact that the primary component of the critical locus moves holomorphically with respect to  $a$  when  $a$  is small [LR] and degenerates uniformly when  $a$  goes to 0 to  $(\mathbb{C} - K_p) \times \mathbb{C}$ .  $\square$

A consequence of Lemma 6 is that the argument of the function  $\alpha|_{\mathbb{S}^1}(\xi)$ , regarded as a function from  $\mathbb{S}^1$  to  $\mathbb{S}^1$ , has degree  $-3$ . This makes the plots of the image of the function  $\alpha|_{\mathbb{S}^1}$  hard to read. The following lemma provides a remedy.

**Proposition 6.2.** *One can choose an appropriate trivialization so that*

$$\lim_{a \rightarrow 0} \frac{\alpha(\xi)}{a} = \frac{1}{2\gamma(\xi)}.$$

**Proof.** Define a trivialization of  $\mathcal{F}_\xi$  that assigns  $c_0(\xi) \rightarrow 0$  and  $c_{-1}(\xi) \rightarrow \gamma^2(\xi)$ , where  $\gamma$  is the Böttcher isomorphism of  $p$ . Note that this is an allowed assignment since it verifies the restrictions in Proposition 5.5.  $\square$

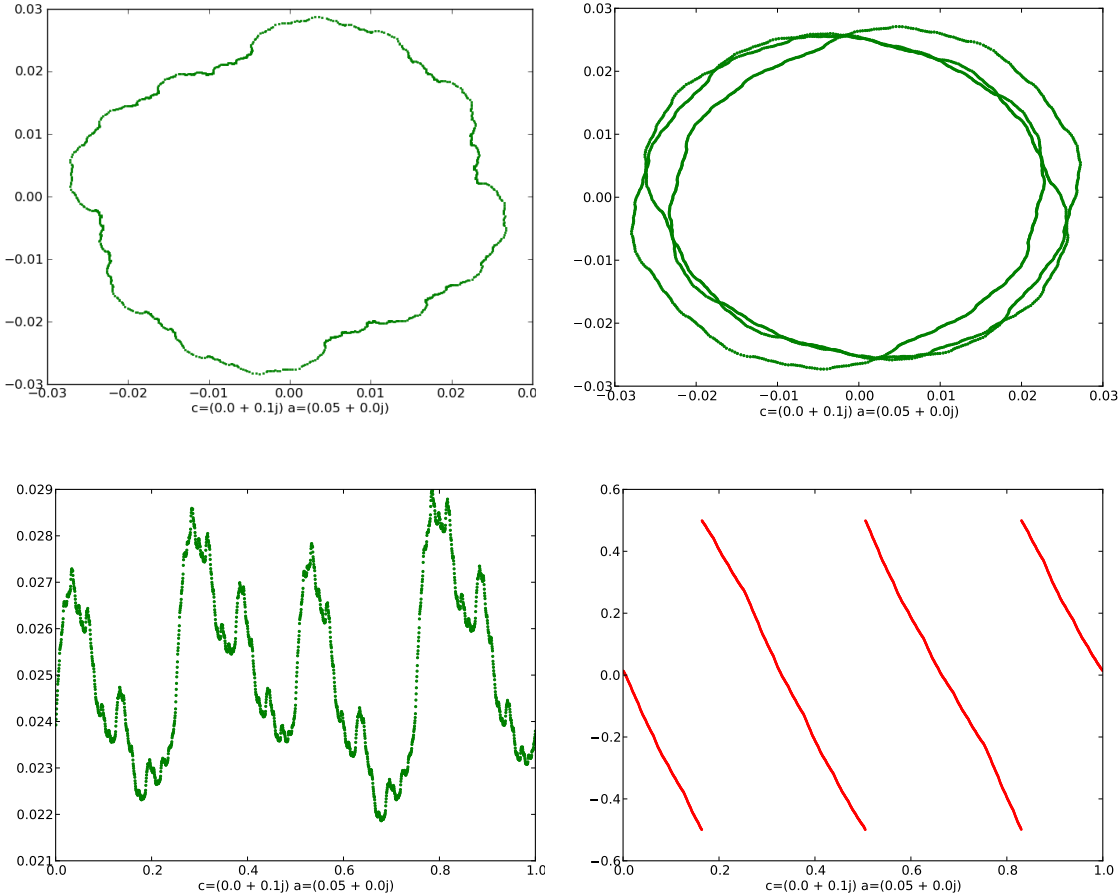
The insight of Lemma 6.2 is that  $\alpha$  measures the contraction induced by the derivative of the Hénon map on the leaves of the lamination of  $J^+ \cup U^+$ , whereas

$$2\gamma(\xi) = p'(\gamma(\xi))$$

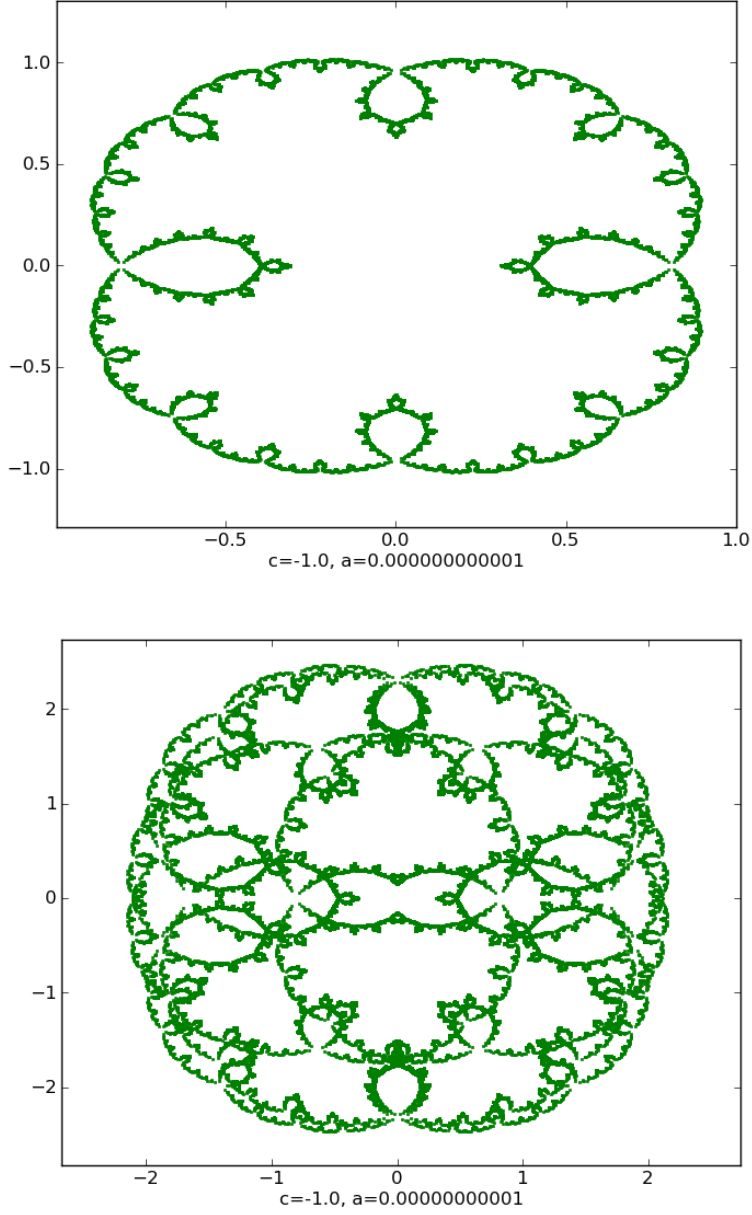
measures the expansion of the polynomial  $p$  on the Julia set  $J_p$  and on  $\mathbb{C} - K_p$ . As the Jacobian  $a$  becomes small, these two quantities behave like the small and respectively the big eigenvalue of the Jacobian  $DH$  of the Hénon map, so their product is close to the determinant  $\det(DH) = a$ .

7. THE IMAGE OF THE COCYCLE  $\alpha$  ON THE UNIT CIRCLE

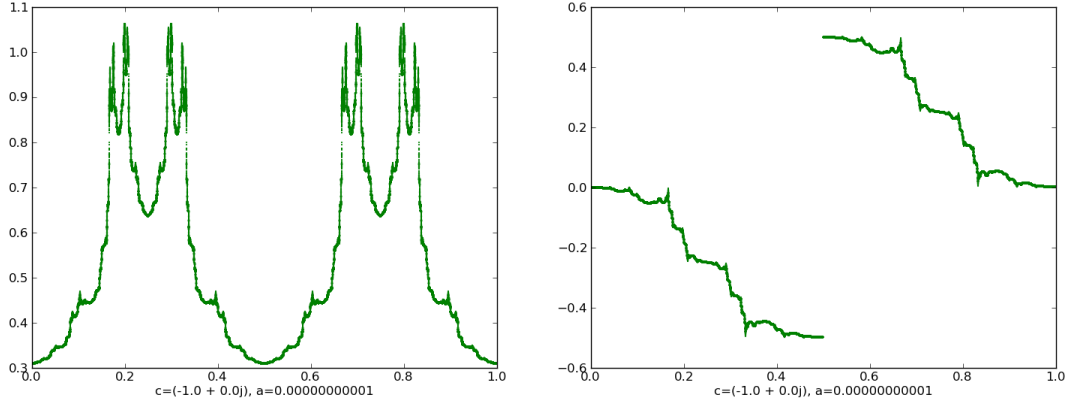
The most interesting behavior of the function  $\alpha$  is on the unit circle  $\mathbb{S}^1$ . We know that  $\alpha : \mathbb{C} - \mathbb{D} \rightarrow \mathbb{C}^*$  is continuous on  $\mathbb{S}^1$ , nonetheless, we expect that  $\alpha$  gives rise to a fractal set when restricted to  $\mathbb{S}^1$ . In [T], we have designed an algorithm in Python for computing the image of  $\alpha$  on the unit circle. Here are some pictures obtained with our program.



**Figure 4.** Pictures for the parameters  $c = 0.1i$  and  $a = 0.05$ . TOP LEFT: The image of  $\alpha$  on  $\mathbb{S}^1$  ( $\alpha$  is computed with respect to the trivialization from Proposition 6.2). TOP RIGHT: The image of  $\alpha$  on  $\mathbb{S}^1$  ( $\alpha$  is computed with respect to the standard trivialization). BOTTOM LEFT: The graph of the absolute value of  $\alpha$  on  $\mathbb{S}^1$  (standard trivialization). BOTTOM RIGHT: The graph of the argument of  $\alpha$  on  $\mathbb{S}^1$  (standard trivialization). The argument is regarded as a function from  $[0, 1]$  to  $[-1/2, 1/2]$ .



**Figure 5.** Here  $c = -1$  and  $a$  is very small. TOP: The image of  $\alpha/a$  on  $\mathbb{S}^1$  with the trivialization from Proposition 6.2. BOTTOM: The image of  $\alpha/a$  on  $\mathbb{S}^1$  with the standard trivialization from Proposition 6.2. Regarded as a function on  $\mathbb{S}^1$ ,  $\alpha$  has degree  $-3$ .



**Figure 6.** Pictures for the parameters  $c = -1$  and  $a = 10^{-10}$ . The function  $\alpha$  has been computed with respect to the trivialization from 6.2. LEFT: The graph of the absolute value of  $\alpha/a$  on  $\mathbb{S}^1$ . RIGHT: The graph of the argument of  $\alpha/a$  on  $\mathbb{S}^1$ . The argument is regarded as a function from  $[0, 1]$  to  $[-1/2, 1/2]$ .

### 8. GROWTH ESTIMATES FOR THE GROUP $\Gamma_{p,a}$

The main motivation for studying the properties of  $\alpha$  on  $\mathbb{S}^1$  is that all other extensions can be expressed as functions of  $\alpha$ . The lift  $\tilde{H}$  of the Hénon map, as well as the elements of the group of deck transforms  $\Gamma_{p,a}$  can be extended to  $\mathbb{S}^1 \times \mathbb{C}$  and we can now recursively compute the elements of the group

$$\Gamma_{p,a} = \left\{ \gamma_{\frac{j}{2^k}} \left( \begin{pmatrix} \xi \\ z \end{pmatrix} \right) = \begin{pmatrix} e^{2\pi i \frac{j}{2^k}} \xi \\ p_{j,k}(\xi)z + q_{j,k}(\xi) \end{pmatrix}, 1 \leq j \leq 2^k \right\} \quad (19)$$

from the relation  $\tilde{H} \circ \gamma_{\frac{j}{2^{k+1}}} = \gamma_{\frac{j}{2^k}} \circ \tilde{H}$  and  $\gamma_1 = \text{id}$  and get

$$\begin{aligned} p_{j,k}(\xi) &= p_{j,k-1}(\xi^2) \frac{\alpha(\xi)}{\alpha \left( e^{2\pi i \frac{j}{2^k}} \xi \right)} \\ q_{j,k}(\xi) &= \frac{p_{j,k-1}(\xi^2)\beta(\xi) + q_{j,k-1}(\xi^2) - \beta \left( e^{2\pi i \frac{j}{2^k}} \xi \right)}{\alpha \left( e^{2\pi i \frac{j}{2^k}} \xi \right)}. \end{aligned}$$

Indeed, we know that  $\tilde{H}(\xi, z) = (\xi^2, \alpha(\xi)z + \beta(\xi))$ ,  $\gamma_1(\xi, z) = (\xi, z)$ , and for  $k > 1$

$$\tilde{H} \circ \begin{pmatrix} e^{2\pi i \frac{j}{2^k}} \xi \\ p_{j,k}(\xi)z + q_{j,k}(\xi) \end{pmatrix} = \begin{pmatrix} e^{2\pi i \frac{j}{2^{k-1}}} \xi \\ p_{j,k-1}(\xi)z + q_{j,k-1}(\xi) \end{pmatrix} \circ \tilde{H}(\xi, z).$$

By comparing the second coordinate we get the following relation

$$\alpha \left( e^{2\pi i \frac{j}{2^k}} \xi \right) \cdot (p_{j,k}(\xi)z + q_{j,k}(\xi)) + \beta(e^{2\pi i \frac{j}{2^k}} \xi) = p_{j,k-1}(\xi^2) (\alpha(\xi)z + \beta(\xi)) + q_{j,k-1}(\xi^2).$$

Therefore

$$\begin{cases} \alpha \left( e^{2\pi i \frac{j}{2^k}} \xi \right) \cdot p_{j,k}(\xi) = p_{j,k-1}(\xi^2) \cdot \alpha(\xi) \\ \alpha \left( e^{2\pi i \frac{j}{2^k}} \xi \right) \cdot q_{j,k}(\xi) + \beta \left( e^{2\pi i \frac{j}{2^k}} \xi \right) = p_{j,k-1}(\xi^2) \cdot \beta(\xi) + q_{j,k-1}(\xi^2) \end{cases}$$

hence we get exactly the description of the group elements from Equation 20.

One can then use the recursive formula to describe each group element  $\gamma_{\frac{j}{2^k}}(\xi, z)$ . Let

$\omega = e^{2\pi i \frac{j}{2^k}}$ . Assume that  $j$  is odd. Otherwise we would need to look at a smaller  $k$ . Notice that the first integer  $m$  for which  $\omega^{2^m} = -1$  is  $m = k - 1$ . We compute:

$$\begin{aligned} p_{j,k}(\xi) &= \frac{\prod_{s=0}^{k-1} \alpha(\xi^{2^s})}{\prod_{s=0}^{k-1} \alpha((\omega\xi)^{2^s})} \\ q_{j,k}(\xi) &= \frac{p_{j,k-1}(\xi^2)\beta(\xi) + q_{j,k-1}(\xi^2) - \beta(\omega\xi)}{\alpha(\omega\xi)} \end{aligned} \quad (20)$$

We choose the standard trivialization as in Proposition 6.1 such that

$$\lim_{a \rightarrow 0} \frac{\alpha(\xi)}{a} = \frac{\gamma(\xi)}{2\gamma^2(\xi^2)}.$$

Then  $\beta(\xi) = -\alpha(\xi)$  and the relation for  $q_{j,k}$  is

$$q_{j,k}(\xi) = \frac{-\sum_{s=0}^{k-1} \prod_{t=s}^{k-1} \alpha(\xi^{2^t}) + \sum_{s=0}^{k-1} \prod_{t=s}^{k-1} \alpha((\omega\xi)^{2^t})}{\prod_{s=0}^{k-1} \alpha((\omega\xi)^{2^s})}. \quad (21)$$

Define for simplicity  $\Pi_s(\xi) = \prod_{t=s}^{k-1} \alpha(\xi^{2^t})$  for  $0 \leq s \leq k-1$ . So  $\Pi_s(\omega\xi) = \prod_{t=s}^{k-1} \alpha((\omega\xi)^{2^t})$  and in particular  $\Pi_{k-1}(\xi) = \alpha(\xi^{2^{k-1}})$  and  $\Pi_{k-1}(\omega\xi) = \alpha(-\xi^{2^{k-1}})$ . The formulas for  $p_{j,k}$  and  $q_{j,k}$  simplify to

$$p_{j,k}(\xi) = \frac{\Pi_0(\xi)}{\Pi_0(\omega\xi)} \quad (22)$$

$$q_{j,k}(\xi) = \frac{\Pi_{k-1}(\omega\xi) - \Pi_{k-1}(\xi)}{\Pi_0(\omega\xi)} + \sum_{s=0}^{k-2} \frac{\Pi_s(\omega\xi) - \Pi_s(\xi)}{\Pi_0(\omega\xi)}. \quad (23)$$

Let  $\delta := \inf_{\xi \in \mathbb{S}^1} |\gamma(\xi)|$ . Suppose  $p(x) = x^2 + c$  is hyperbolic with connected Julia set  $J_p$  and assume  $\gamma : \mathbb{S}^1 \rightarrow J_p$  is the Carathéodory loop of  $p$ . The critical point  $x = 0$  is in the interior of the filled-in Julia set  $K_p$  so

$$0 < \delta \leq |\gamma(\xi)| \leq 2.$$

Moreover  $p(\gamma(\xi)) = \gamma(\xi^2)$  and  $p(\gamma(-\xi)) = \gamma(\xi^2)$ . This gives  $\gamma(\xi)^2 = \gamma(-\xi)^2$  and  $\gamma(-\xi) = -\gamma(\xi)$ . Note that  $\gamma(\xi)$  is not equal to  $\gamma(-\xi)$  since otherwise the external rays corresponding to  $\xi$  and  $-\xi$  land at the same point  $\gamma(\xi) \in J_p$  and they are mapped under

$p$  to the same external ray landing at  $\gamma(\xi^2)$ . This is possible only if  $\gamma(\xi) = 0$ , the critical point, which is a contradiction since  $0 \in \overset{\circ}{K}_p$ .

**Lemma 8.1.** *There exists  $\delta' > 0$  such that for all  $a$  with  $|a| < \delta'$ , we have*

$$\left| \frac{\alpha(\xi)}{a} \right| < \frac{2}{\delta^2},$$

for all  $\xi \in \mathbb{S}^1$ .

**Proof.** We have that

$$\lim_{a \rightarrow 0} \left| \frac{\alpha(\xi)}{a} \right| = \left| \frac{\gamma(\xi)}{2\gamma^2(\xi^2)} \right|.$$

Fix  $\epsilon = 1/\delta^2 > 0$ . Then there exists  $\delta' > 0$  such that for all  $|a| < \delta'$ ,

$$\left| \left| \frac{\alpha(\xi)}{a} \right| - \frac{|\gamma(\xi)|}{2|\gamma(\xi^2)|^2} \right| < \epsilon,$$

and in particular

$$\left| \frac{\alpha(\xi)}{a} \right| < \epsilon + \frac{|\gamma(\xi)|}{2|\gamma(\xi^2)|^2} \leq \frac{1}{\delta^2} + \frac{2}{2\delta^2} = \frac{2}{\delta^2}.$$

□

**Lemma 8.2.** *There exists  $\delta'' > 0$  such that for all  $a$  with  $|a| < \delta''$ , we have*

$$\left| \frac{\alpha(\xi)}{a} - \frac{\alpha(-\xi)}{a} \right| > \frac{\delta}{8},$$

for all  $\xi \in \mathbb{S}^1$ .

**Proof.** We have

$$\lim_{a \rightarrow 0} \left| \frac{\alpha(\xi)}{a} - \frac{\alpha(-\xi)}{a} \right| = \left| \frac{\gamma(\xi)}{2\gamma^2(\xi^2)} - \frac{\gamma(-\xi)}{2\gamma^2(\xi^2)} \right| = \left| \frac{\gamma(\xi)}{\gamma^2(\xi^2)} \right|,$$

since  $\gamma(-\xi) = -\gamma(\xi)$ . Fix  $\epsilon = \delta/8 > 0$ . There exists  $\delta'' > 0$  such that for all  $|a| < \delta''$ ,

$$\left| \left| \frac{\alpha(\xi)}{a} - \frac{\alpha(-\xi)}{a} \right| - \frac{|\gamma(\xi)|}{|\gamma(\xi^2)|^2} \right| < \epsilon,$$

and in particular

$$\left| \frac{\alpha(\xi)}{a} - \frac{\alpha(-\xi)}{a} \right| > -\epsilon + \frac{|\gamma(\xi)|}{|\gamma(\xi^2)|^2} \geq -\frac{\delta}{8} + \frac{\delta}{4} = \frac{\delta}{8}.$$

□

**Proposition 8.3 (Growth estimate).** *Suppose  $j$  is odd. There exists  $a_0 > 0$  such that for all  $0 < |a| < a_0$  there exists a positive integer  $k_0$  such that for all  $k \geq k_0$*

$$|p_{j,k}(\xi)z + q_{j,k}(\xi)| > \frac{\delta^3}{32} \left( \frac{\delta^2}{2|a|} \right)^{k-1} - |z|.$$

The integer  $k_0$  depends only on  $a_0$  and  $z$ .

**Proof.** From Equations 23 and 22 we get

$$\begin{aligned}
|p_{j,k}(\xi)z + q_{j,k}(\xi)| &= \left| \frac{\Pi_{k-1}(\omega\xi) - \Pi_{k-1}(\xi)}{\Pi_0(\omega\xi)} + \sum_{s=0}^{k-2} \frac{\Pi_s(\omega\xi) - \Pi_s(\xi)}{\Pi_0(\omega\xi)} + \frac{\Pi_0(\xi)}{\Pi_0(\omega\xi)} z \right| \\
&\geq \left| \frac{\Pi_{k-1}(\omega\xi) - \Pi_{k-1}(\xi)}{\Pi_0(\omega\xi)} \right| - \left| \sum_{s=0}^{k-2} \frac{\Pi_s(\omega\xi) - \Pi_s(\xi)}{\Pi_0(\omega\xi)} + \frac{\Pi_0(\xi)}{\Pi_0(\omega\xi)} z \right| \\
&\geq \frac{|\Pi_{k-1}(\omega\xi) - \Pi_{k-1}(\xi)|}{|\Pi_0(\omega\xi)|} - \sum_{s=0}^{k-2} \frac{|\Pi_s(\xi)| + |\Pi_s(\omega\xi)|}{|\Pi_0(\omega\xi)|} - \frac{|\Pi_0(\xi)|}{|\Pi_0(\omega\xi)|} |z|
\end{aligned}$$

**The leading term.**

$$\begin{aligned}
\frac{|\Pi_{k-1}(\omega\xi) - \Pi_{k-1}(\xi)|}{|\Pi_0(\omega\xi)|} &= \frac{\left| \alpha(-\xi^{2^{k-1}}) - \alpha(\xi^{2^{k-1}}) \right|}{\prod_{t=0}^{k-1} \left| \alpha((\omega\xi)^{2^t}) \right|} = \frac{1}{|a|^{k-1}} \cdot \frac{\left| \frac{\alpha(-\xi^{2^{k-1}})}{a} - \frac{\alpha(\xi^{2^{k-1}})}{a} \right|}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|} \\
&\geq \frac{\frac{\delta}{8}}{|a|^{k-1}} \cdot \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|}
\end{aligned}$$

**The  $s$ -term.**

$$\begin{aligned}
\frac{|\Pi_s(\xi)|}{|\Pi_0(\omega\xi)|} &= \frac{\left| \prod_{t=s}^{k-1} \alpha(\xi^{2^t}) \right|}{\left| \prod_{t=0}^{k-1} \alpha((\omega\xi)^{2^t}) \right|} = \frac{\prod_{t=s}^{k-1} \left| \frac{\alpha(\xi^{2^t})}{a} \right| |a|^{k-s}}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right| |a|^k} = \frac{1}{|a|^s} \cdot \frac{\prod_{t=s}^{k-1} \left| \frac{\alpha(\xi^{2^t})}{a} \right|}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|} \\
&\leq \frac{1}{|a|^s} \cdot \frac{\left( \frac{2}{\delta^2} \right)^{k-s}}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|} = \left( \frac{2}{\delta^2} \right)^k \cdot \left( \frac{\delta^2}{2|a|} \right)^s \cdot \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|}
\end{aligned}$$

Similarly we can show that

$$\frac{|\Pi_s(\omega\xi)|}{|\Pi_0(\omega\xi)|} \leq \left( \frac{2}{\delta^2} \right)^k \cdot \left( \frac{\delta^2}{2|a|} \right)^s \cdot \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|}$$

Putting together all inequalities we get that

$$|p_{j,k}(\xi)z + q_{j,k}(\xi)| \geq \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|} \left( \frac{\delta}{8|a|^{k-1}} - 2 \sum_{s=0}^{k-2} \left( \frac{2}{\delta^2} \right)^k \left( \frac{\delta^2}{2|a|} \right)^s - \left( \frac{2}{\delta^2} \right)^k |z| \right).$$



We can compute explicitly the sum in the middle and get

$$\begin{aligned}
\sum_{s=0}^{k-2} \left(\frac{2}{\delta^2}\right)^k \left(\frac{\delta^2}{2|a|}\right)^s &= \left(\frac{2}{\delta^2}\right)^k \sum_{s=0}^{k-2} \left(\frac{\delta^2}{2|a|}\right)^s = \left(\frac{2}{\delta^2}\right)^k \frac{\left(\frac{\delta^2}{2|a|}\right)^{k-1} - 1}{\frac{\delta^2}{2|a|} - 1} \\
&= \frac{2}{\delta^2} \cdot \frac{2|a|}{\delta^2 - 2|a|} \cdot \frac{1}{|a|^{k-1}} - \frac{2|a|}{\delta^2 - 2|a|} \left(\frac{2}{\delta^2}\right)^k \\
&= C_1(a) \frac{1}{|a|^{k-1}} - C_2(a) \left(\frac{2}{\delta^2}\right)^k,
\end{aligned}$$

where  $C_1(a) := \frac{2}{\delta^2} \frac{2|a|}{\delta^2 - 2|a|}$  and  $C_2(a) := \frac{2|a|}{\delta^2 - 2|a|}$  are constants that depend on  $a$ . We get

$$\begin{aligned}
|p_{j,k}(\xi)z + q_{j,k}(\xi)| &\geq \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|} \left( \left(\frac{\delta}{8} - 2C_1(a)\right) \frac{1}{|a|^{k-1}} + (2C_2(a) - |z|) \left(\frac{2}{\delta^2}\right)^k \right) \\
&> \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|} \left( \left(\frac{\delta}{8} - 2C_1(a)\right) \frac{1}{|a|^{k-1}} - |z| \left(\frac{2}{\delta^2}\right)^k \right),
\end{aligned}$$

since  $C_2(a)$  is positive. Note that  $C_1(a)$  can be made arbitrary small. In particular, if  $|a| < \frac{\delta^2}{2} \cdot \frac{\delta^3}{\delta^3 + 64}$  then  $\frac{\delta}{8} - 2C_1(a) > \frac{\delta}{16}$ . To see this, notice that

$$\frac{2|a|}{\delta^2} < \frac{\delta^3}{\delta^3 + 64} \quad \text{and so} \quad \frac{\delta^2}{2|a|} - 1 > \frac{64 + \delta^3}{\delta^3} - 1 = \frac{64}{\delta^3}.$$

Then

$$C_1(a) = \frac{2}{\delta^2} \cdot \frac{1}{\frac{\delta^2}{2|a|} - 1} < \frac{2}{\delta^2} \frac{\delta^3}{64} = \frac{\delta}{32}.$$

and  $\frac{\delta}{8} - 2C_1(a) > \frac{\delta}{8} - \frac{2\delta}{32} = \frac{\delta}{16}$ . Under this assumption we have shown that

$$|p_{j,k}(\xi)z + q_{j,k}(\xi)| > \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right|} \left( \frac{\delta}{16} \frac{1}{|a|^{k-1}} - |z| \left(\frac{2}{\delta^2}\right)^k \right).$$

If  $|a| < \frac{\delta^2}{2} \cdot \frac{\delta^3}{\delta^3 + 64}$  then  $\frac{\delta^2}{2|a|} > \frac{\delta^3 + 64}{\delta^3} > 1$ . Thus there exists an integer  $k_0$  such that for all  $k \geq k_0$  we have

$$\frac{\delta}{16} \frac{1}{|a|^{k-1}} - |z| \left(\frac{2}{\delta^2}\right)^k > 0 \Leftrightarrow \left(\frac{\delta^2}{2|a|}\right)^{k-1} > \frac{32}{\delta^3} |z|. \quad (24)$$

In view of Lemma 8.1 we get

$$\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2^t})}{a} \right| \leq \left(\frac{\delta^2}{2}\right)^{-k}.$$

Hence, for  $k \geq k_0$  we have

$$\begin{aligned} |p_{j,k}(\xi)z + q_{j,k}(\xi)| &> \frac{1}{\prod_{t=0}^{k-1} \left| \frac{\alpha((\omega\xi)^{2t})}{a} \right|} \left( \frac{\delta}{16} \frac{1}{|a|^{k-1}} - |z| \left( \frac{2}{\delta^2} \right)^k \right) \\ &> \left( \frac{\delta^2}{2} \right)^k \left( \frac{\delta}{16} \frac{1}{|a|^{k-1}} - |z| \left( \frac{2}{\delta^2} \right)^k \right) = \frac{\delta^3}{32} \left( \frac{\delta^2}{2|a|} \right)^{k-1} - |z| \end{aligned}$$

The constant  $a_0 > 0$  can be taken to be

$$a_0 = \min \left( \delta', \delta'', \frac{\delta^2}{2} \frac{\delta^3}{\delta^3 + 64} \right),$$

where  $\delta' > 0$  and  $\delta'' > 0$  are the same constants from Lemmas 8.1 and 8.2.  $\square$

**Proposition 8.4.** *Let  $(\xi_0, z_0) \in \mathbb{S}^1 \times \mathbb{C}$ . There exists a neighborhood  $U \subset \mathbb{S}^1 \times \mathbb{C}$  of  $(\xi_0, z_0)$  such that*

$$\gamma_{\frac{j}{2^k}}(U) \cap U = \emptyset$$

for all elements  $\gamma_{\frac{j}{2^k}} \in \Gamma_{p,a}$ , with  $\gamma_{\frac{j}{2^k}} \neq id$ .

**Proof.** Fix  $a$  with  $|a| < a_0$  as in the proof of Proposition 8.3. Consider a neighborhood  $U_0 \subset \mathbb{C}$  of  $z_0$  defined by  $|z - z_0| < \frac{\delta^3}{32}$ . Then for all  $z \in U_0$  we have  $|z| < |z_0| + \frac{\delta^3}{32}$ . There exists a smallest positive integer  $k_0$  which is large enough so that for all  $k \geq k_0$  the following inequality holds

$$\left( \frac{\delta^2}{2|a|} \right)^{k-1} > \frac{64}{\delta^3} \left( |z_0| + \frac{\delta^3}{32} \right) > \frac{32}{\delta^3} |z|,$$

for all  $z \in U_0$ . Hence Condition 24 in the proof of Proposition 8.3 is satisfied for all  $z \in U_0$ . For  $j$  odd and  $k \geq k_0$  we get

$$\begin{aligned} |p_{j,k}(\xi)z + q_{j,k}(\xi) - z_0| &\geq |p_{j,k}(\xi)z + q_{j,k}(\xi)| - |z_0| > \frac{\delta^3}{32} \left( \frac{\delta^2}{2|a|} \right)^{k-1} - |z| - |z_0| \\ &> \frac{\delta^3}{32} \left( \frac{\delta^2}{2|a|} \right)^{k-1} - 2|z_0| - \frac{\delta^3}{32} \\ &> \frac{\delta^3}{32} \frac{64}{\delta^3} \left( |z_0| + \frac{\delta^3}{32} \right) - 2|z_0| - \frac{\delta^3}{32} = \frac{\delta^3}{32}. \end{aligned}$$

It follows that  $|p_{j,k}(\xi)z + q_{j,k}(\xi) - z_0| > \frac{\delta^3}{32}$  for all  $z \in U_0$  and for all  $k \geq k_0$  and  $j$  odd. If  $j$  is even then the situation is similar by considering a smaller  $k$ . Suppose  $k_0 > 2$ . If  $1 < k < k_0$  then we look at the first component of  $\gamma_{\frac{j}{2^k}}(\xi, z)$  where

$$\gamma_{\frac{j}{2^k}} \begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} e^{2\pi i \frac{j}{2^k}} \xi \\ p_{j,k}(\xi)z + q_{j,k}(\xi) \end{pmatrix}.$$

Set  $V_0 = \{\xi \in \mathbb{S}^1 : |\arg(\xi) - \arg(\xi_0)| < 1/2^{k_0}\}$  and let  $U = V_0 \times U_0$  be a neighborhood of  $(\xi_0, z_0)$ . Then

$$\gamma_{\frac{j}{2^k}}(U) \cap U = \emptyset \quad \text{for all } j, k \text{ with } \frac{j}{2^k} \neq 1.$$

To summarize, when  $k$  is large (i.e.  $k \geq k_0$ ) the second component of  $\gamma_{\frac{j}{2^k}}(\xi, z)$  exits  $U$  and when  $k$  is small, the first component of  $\gamma_{\frac{j}{2^k}}(\xi, z)$  exits  $U$ .  $\square$

## 9. MAIN RESULTS

The growth estimates described in Section 8 provide a powerful tool for analyzing the properties of the extension of the group of deck transforms  $\Gamma_{p,a}$  to the boundary of the covering manifold. We first recall some basic properties of group actions.

**Definition 9.1.** Let  $X$  be a locally compact metric space. A discrete group  $G$  acts properly discontinuously on  $X$  if for every  $x \in X$  there exists a neighborhood  $U \subset X$  of  $x$  such that  $g(U) \cap U = \emptyset$  for all group elements  $g \in G$ ,  $g \neq id$ .

**Definition 9.2.** Let  $X$  be a locally compact metric space. A discrete group  $G$  acts freely on  $X$  if for every  $x \in X$   $g(x) \neq x$  for all group elements  $g \in G$ ,  $g \neq id$ .

We are now able to prove the first theorem about the action of the group  $\Gamma_{p,a}$  on  $\mathbb{S}^1 \times \mathbb{C}$ , the boundary of  $(\mathbb{C} - \mathbb{D}) \times \mathbb{C}$ .

**Theorem 9.3.** *Let  $p$  be a hyperbolic quadratic polynomial with connected Julia set. There exists  $a_0 > 0$  such that for all  $a$  with  $0 < |a| < a_0$  the group  $\Gamma_{p,a}$  acts freely and properly discontinuously on  $\mathbb{S}^1 \times \mathbb{C}$ .*

**Proof.** It is easy to see that the action of  $\Gamma_{p,a}$  is free on  $\mathbb{S}^1 \times \mathbb{C}$ . Take any element  $\gamma_{\frac{j}{2^k}} \in \Gamma_{p,a}$ ,  $\gamma_{\frac{j}{2^k}} \neq id$ . Then

$$\gamma_{\frac{j}{2^k}}(\xi, z) = (\omega\xi, p_{j,k}(\xi)z + q_{j,k}(\xi)), \quad \text{where } \omega = e^{2\pi i \frac{j}{2^k}} \neq 1.$$

So the only group element that fixes the first component is the identity. The fact that the action of the group on  $\mathbb{S}^1 \times \mathbb{C}$  is properly discontinuous is the hard part of the theorem and it follows from Proposition 8.4, which in turn uses the growth estimates proved in Proposition 8.3 in an essential way.  $\square$

**Corollary 9.3.1.**  $\mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a}$  and  $(\mathbb{C} - \mathbb{D}) \times \mathbb{C}/\Gamma_{p,a}$  are topological manifolds, with fundamental group  $\mathbb{Z}[1/2]/\mathbb{Z}$ .

**Proof.** By Theorem 3.6,  $(\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}$  is a covering space of  $U^+$ , so the action of  $\Gamma_{p,a}$  on  $(\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}$  is properly discontinuous and without fixed points. By Theorem 9.3, the action of  $\Gamma_{p,a}$  on  $\mathbb{S}^1 \times \mathbb{C}$  is also properly discontinuous and without fixed points.  $\square$

In general, it would be interesting to study whether the group always acts properly discontinuous on  $\mathbb{S}^1 \times \mathbb{C}$  as in Theorem 9.3 or whether there are examples of Hénon maps for which the group has limit sets on  $\mathbb{S}^1 \times \mathbb{C}$ .

**Theorem 9.4.** *Let  $p$  be a hyperbolic quadratic polynomial with connected Julia set. There exists  $a_0 > 0$  such that for all  $a$  with  $0 < |a| < a_0$  the following hold*

- (a) *There exists a continuous surjective map  $\hat{\pi}$ , holomorphic on the leaves of the foliation of  $J^+$ , that makes the following diagram commute*

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a} & \xrightarrow{\tilde{H}} & \mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a} \\ \hat{\pi} \downarrow & & \downarrow \hat{\pi} \\ J^+ & \xrightarrow{H} & J^+ \end{array}$$

- (b) *There exists a continuous surjective map  $\hat{\pi}$ , biholomorphic on  $(\mathbb{C} - \mathbb{D}) \times \mathbb{C}/\Gamma_{p,a}$  and holomorphic on the leaves of the foliation of  $J^+$ , that makes the diagram commute*

$$\begin{array}{ccc} (\mathbb{C} - \mathbb{D}) \times \mathbb{C}/\Gamma_{p,a} & \xrightarrow{\tilde{H}} & (\mathbb{C} - \mathbb{D}) \times \mathbb{C}/\Gamma_{p,a} \\ \hat{\pi} \downarrow & & \downarrow \hat{\pi} \\ \overline{U}^+ & \xrightarrow{H} & \overline{U}^+ \end{array}$$

**Proof.** By Equation 19, the map  $\tilde{H} : (\mathbb{C} - \mathbb{D}) \times \mathbb{C} \rightarrow (\mathbb{C} - \mathbb{D}) \times \mathbb{C}$  satisfies the relation

$$\tilde{H} \circ \gamma_{\frac{j}{2^{k+1}}} = \gamma_{\frac{j}{2^k}} \circ \tilde{H} \text{ and } \gamma_1 = \text{id}.$$

In Lemma 5.1, we constructed the function  $\pi : (\mathbb{C} - \mathbb{D}) \times \mathbb{C} \rightarrow U^+ \cup J^+$ , with the property that  $\pi \circ \tilde{H} = H \circ \pi$ . Since  $\Gamma_{p,a}$  is a group of deck transforms, we have  $\pi \circ \gamma = \pi$  for any  $\gamma \in \Gamma_{p,a}$ . So both  $\tilde{H}$  and  $\pi$  descend to the quotient  $(\mathbb{C} - \mathbb{D}) \times \mathbb{C}/\Gamma_{p,a}$ . By abuse of notation, we will still use  $\tilde{H}$  in place of  $\hat{\tilde{H}}$ . The map  $\hat{\pi}$  is a continuous surjection, holomorphic on the leaves on the foliation of  $J^+$  and  $U^+$ , and  $\hat{\pi} : (\mathbb{C} - \mathbb{D}) \times \mathbb{C}/\Gamma_{p,a} \rightarrow U^+$  is injective, by Theorem 3.6.  $\square$

**Remark 9.5.** Part (a) of Theorem 9.4 can be viewed as a two dimensional analog of the Carathéodory loop from one dimensional dynamics. The universal object in this case is not the circle  $\mathbb{S}^1$ , but rather a 3-dimensional topological manifold, isomorphic to a quotient of  $\mathbb{S}^1 \times \mathbb{C}$  by a discrete group action.

**Corollary 9.5.1.** *Let  $p$  be a quadratic polynomial with an attractive fixed point. There exists  $a_0 > 0$  such that for all  $0 < |a| < a_0$  the closure of the escaping set  $U^+$  of the Hénon map  $H_{p,a}$  satisfies  $\overline{U}^+ \simeq (\mathbb{C} - \mathbb{D}) \times \mathbb{C}/\Gamma_{p,a}$ . The Julia set  $J^+$  is a topological manifold and  $J^+ \simeq \mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a}$ .*

**Proof.** By Theorem 2.3, the boundary of the primary component is homeomorphic to  $\mathbb{S}^1$ . The projection  $\hat{\pi} : \mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a} \rightarrow J^+$  from Theorem 9.4 is bijective.  $\square$

In one-dimensional dynamics, W. Thurston [Th] has constructed topological models for the Julia sets of quadratic polynomials as quotients of the unit circle. Consider a hyperbolic polynomial  $p(x) = x^2 + c$  with connected Julia set  $J_p$  and let  $\gamma : \mathbb{S}^1 \rightarrow J_p$  be the Carathéodory loop of  $p$ . Thurston defined an equivalence relation on  $\mathbb{S}^1$  using

the Carathéodory loop,  $\xi_1 \sim \xi_2$  whenever  $\gamma(\xi_1) = \gamma(\xi_2)$ , and showed that  $\mathbb{S}^1/\sim$  is homeomorphic to the Julia set  $J_p$ .

Similarly, we will introduce an equivalence relation on  $\mathbb{S}^1 \times \mathbb{C}$ , and respectively on  $\mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a}$  for  $a$  small. When there is no confusion, we will denote the group  $\Gamma_{p,a}$  by  $\Gamma$  and the orbit of a point  $(\xi, z)$  under the group  $\Gamma_{p,a}$  by

$$\mathcal{O}_\Gamma((\xi, z)) = \left\{ \gamma_{\frac{j}{2^k}}(\xi, z) : k \geq 0, 1 \leq j \leq 2^k \right\}.$$

**Definition 9.6 (Equivalence of points).** Let  $\xi_1, \xi_2 \in \mathbb{S}^1$  and  $z \in \mathbb{C}$ . We will say that

$$(\xi_1, z) \sim_p (\xi_2, z) \text{ if } \gamma(\xi_1) = \gamma(\xi_2).$$

The following elementary proposition will be useful.

**Proposition 9.7.** *Let  $\xi_1, \xi_2 \in \mathbb{S}^1$  such that  $\gamma(\xi_1) = \gamma(\xi_2)$ . Let  $\omega_1 = e^{2\pi i \frac{j}{2^k}}$  be a dyadic root of unity, where  $j$  is odd. There exists  $m$  odd such that if we set  $\omega_2 = e^{2\pi i \frac{m}{2^k}}$ , then  $\gamma(\omega_1 \xi_1) = \gamma(\omega_2 \xi_2)$ .*

**Proof.** By induction on  $k$ . We use the fact that  $p(\gamma(\xi)) = \gamma(\xi^2)$ , for any  $\xi \in \mathbb{S}^1$ .  $\square$

**Proposition 9.8.** *Let  $\xi, \xi_2 \in \mathbb{S}^1$  and  $z \in \mathbb{C}$  such that  $(\xi_1, z) \sim_p (\xi_2, z)$ . Let  $k$  be a non-negative integer and  $j$  an odd number with  $1 \leq j \leq 2^k$ . There exists  $m$  odd,  $1 \leq m \leq 2^k$  such that*

$$\gamma_{\frac{j}{2^k}}(\xi_1, z) \sim_p \gamma_{\frac{m}{2^k}}(\xi_2, z).$$

**Proof.** Let  $\omega_1 = e^{2\pi i \frac{j}{2^k}}$ . By Lemma 9.7, there exists  $m$  odd such that, if  $\omega_2 = e^{2\pi i \frac{m}{2^k}}$  then  $\gamma(\omega_1 \xi_1) = \gamma(\omega_2 \xi_2)$ . We will look at the group elements

$$\gamma_{\frac{j}{2^k}} \begin{pmatrix} \xi_1 \\ z \end{pmatrix} = \begin{pmatrix} \omega_1 \xi_1 \\ p_{j,k}(\xi_1)z + q_{j,k}(\xi_1) \end{pmatrix} \quad \text{and} \quad \gamma_{\frac{m}{2^k}} \begin{pmatrix} \xi_2 \\ z \end{pmatrix} = \begin{pmatrix} \omega_2 \xi_2 \\ p_{m,k}(\xi_2)z + q_{m,k}(\xi_2) \end{pmatrix}.$$

Using Equations 20 and 21, and Lemma 5.8 we get that

$$p_{j,k}(\xi_1)z + q_{j,k}(\xi_1) = p_{m,k}(\xi_2)z + q_{m,k}(\xi_2),$$

so the result follows.  $\square$

**Corollary 9.8.1.** Let  $(\xi_1, z), (\xi_2, z)$  be two points in  $\mathbb{S}^1 \times \mathbb{C}$  such that  $(\xi_1, z) \sim_p (\xi_2, z)$ . Then any point in  $\mathcal{O}_\Gamma((\xi_1, z))$  is  $\sim_p$  equivalent to some other point in  $\mathcal{O}_\Gamma((\xi_2, z))$ .

**Remark 9.9.** It is not in general true that if  $(\xi_1, z) \sim_p (\xi_2, z)$  and  $\gamma_{\frac{j}{2^k}}$  is an element of the group  $\Gamma$ , then  $\gamma_{\frac{j}{2^k}}(\xi_1, z) \sim_p \gamma_{\frac{j}{2^k}}(\xi_2, z)$ , so one cannot extend canonically the action of the group  $\Gamma$  to the space  $\mathbb{S}^1 \times \mathbb{C}/\sim_p$ . However, by Corollary 9.8.1, the orbits of  $\mathbb{S}^1 \times \mathbb{C}$  under the group  $\Gamma$  preserve the equivalence relation  $\sim_p$ .

One can extend the notion of  $\sim_p$  equivalence to group orbits as follows:

**Definition 9.10 (Equivalence of orbits).** Let  $(\xi, z), (\xi', z') \in \mathbb{S}^1 \times \mathbb{C}$ . We say that

$$\mathcal{O}_\Gamma((\xi, z)) \sim_p \mathcal{O}_\Gamma((\xi', z'))$$

if there exists  $(\xi'', z) \in \mathcal{O}_\Gamma((\xi', z'))$  such that  $(\xi, z) \sim_p (\xi'', z)$ .

We need to prove first that  $\sim_p$  is well defined. Let  $(t, y)$  be another point in  $\mathcal{O}_\Gamma((\xi, z))$ ,

$$(t, y) = \gamma_{\frac{j}{2^k}}(\xi, z), \text{ for some } k \geq 0 \text{ and } 1 \leq j \leq 2^k, j \text{ odd.}$$

We will show that one can find  $(t', y)$  in  $\mathcal{O}_\Gamma((\xi', z'))$  such that  $(t, y) \sim_p (t', y)$ . We know that there exists  $(\xi'', z)$  in  $\mathcal{O}_\Gamma((\xi', z'))$  such that  $(\xi, z) \sim_p (\xi'', z)$ . By Lemma 9.8, there exists  $m$  odd,  $1 \leq m \leq 2^k$ , such that  $\gamma_{\frac{j}{2^k}}(\xi, z) \sim_p \gamma_{\frac{m}{2^k}}(\xi'', z)$ . Then  $(t', y) = \gamma_{\frac{m}{2^k}}(\xi'', z)$  is the element that we want.

Let us show that  $\sim_p$  is an equivalence relation. The fact that  $\sim_p$  is reflexive and transitive is obvious, so we only show symmetry.

The symmetry property follows almost directly from Lemma 9.8. Suppose that  $\mathcal{O}_\Gamma((\xi, z)) \sim_p \mathcal{O}_\Gamma((\xi', z'))$ . There exists  $(\xi'', z)$  in  $\mathcal{O}_\Gamma((\xi', z'))$ ,  $(\xi'', z) = \gamma_{\frac{j}{2^k}}(\xi', z')$  such that  $(\xi, z) \sim_p (\xi'', z)$ . Then  $(\xi', z') = \gamma_{\frac{2^k-j}{2^k}}(\xi'', z)$ , so by Lemma 9.8, there exists  $m$  odd,  $1 \leq m \leq 2^k$ , such that  $\gamma_{\frac{2^k-j}{2^k}}(\xi'', z) \sim_p \gamma_{\frac{m}{2^k}}(\xi, z)$ . Therefore  $\mathcal{O}_\Gamma((\xi', z')) \sim_p \mathcal{O}_\Gamma((\xi, z))$ , which shows that  $\sim_p$  is symmetric.

**Theorem 9.11.** *Let  $p$  be a hyperbolic quadratic polynomial with connected Julia set  $J_p$ . Let  $\mathcal{M}_{p,a} := \mathbb{S}^1 \times \mathbb{C} / \Gamma_{p,a}$ . There exists  $a_0 > 0$  such that for all  $a$  with  $0 < |a| < a_0$  there exists a conjugacy  $\hat{\pi}$  which makes the following diagram commutative*

$$\begin{array}{ccc} \mathcal{M}_{p,a} / \sim_p & \xrightarrow{\hat{H}} & \mathcal{M}_{p,a} / \sim_p \\ \hat{\pi} \downarrow & & \downarrow \hat{\pi} \\ J^+ & \xrightarrow{H} & J^+ \end{array}$$

where  $\sim_p$  is the equivalence relation from Definition 9.10.

**Proof.** The function  $\hat{\pi} : \mathbb{S}^1 \times \mathbb{C} / \Gamma_{p,a} \rightarrow J^+$  from Theorem 9.4 is a continuous surjection, biholomorphic on the leaves of the lamination of  $J^+$ . If the polynomial from which we perturb is  $p(x) = x^2 + c$  and the parameter  $c$  is not chosen from the interior of the main cardioid of the Mandelbrot set, then the projection  $\hat{\pi}$  is not yet injective.

The projection function  $\pi : \mathbb{S}^1 \times \mathbb{C} \rightarrow J^+$  was first constructed in Lemma 5.1. We defined  $\pi(\xi, z)$  as  $\pi_\xi(z)$ , where  $\pi_\xi$  is the unique biholomorphic map from  $\mathbb{C}$  into  $\mathcal{F}_\xi$  with the property that  $\pi_\xi(0) = c_0(\xi)$  and  $\pi_\xi(1) = c_{-1}(\xi)$ . We show that if  $\pi(\xi_1, z_1) = \pi(\xi_2, z_2)$  then  $\mathcal{O}_\Gamma(\xi_1, z_1) \sim_p \mathcal{O}_\Gamma(\xi_2, z_2)$ .

Assume therefore that  $\pi(\xi_1, z_1) = \pi(\xi_2, z_2)$  for some points  $(\xi_1, z_1)$  and  $(\xi_2, z_2)$  from  $\mathbb{S}^1 \times \mathbb{C}$ . Then  $\mathcal{F}_{\xi_1}$  and  $\mathcal{F}_{\xi_2}$  represent the same leaf of the lamination of  $J^+$  and the functions  $\pi_{\xi_1} : \mathbb{C} \rightarrow \mathcal{F}_{\xi_1}$  and  $\pi_{\xi_2} : \mathbb{C} \rightarrow \mathcal{F}_{\xi_2}$  are potentially different parametrizations of the same leaf. The primary component of the critical locus intersects  $\mathcal{F}_{\xi_2}$  at the points  $c_0(\omega \xi_2)$ , where  $\omega$  is a dyadic root of unity. There exists  $\omega = e^{2\pi i \frac{j}{2^k}}$  some dyadic root of

unity such that

$$c_0(\xi_1) = c_0(\omega\xi_2). \quad (25)$$

However, the identifications of the primary component of the critical locus are completely described in Theorem 2.3, namely we have

$$c_0(\zeta_1) = c_0(\zeta_2) \text{ for } \zeta_1, \zeta_2 \in \mathbb{S}^1 \Leftrightarrow \gamma(\zeta_1) = \gamma(\zeta_2). \quad (26)$$

From Relations 25 and 26 it follows that  $\gamma(\xi_1) = \gamma(\omega\xi_2)$ . The Carathéodory loop  $\gamma$  verifies the conjugacy relation  $\gamma(\xi^2) = p(\gamma(\xi))$  so we must also have  $\gamma(\xi_1^2) = \gamma(\omega^2\xi_2^2)$ . Therefore, by Relation 26, the following equality holds true

$$c_{-1}(\xi_1) = H^{-1}(c_0(\xi_1^2)) = H^{-1}(c_0(\omega^2\xi_2^2)) = c_{-1}(\omega\xi_2).$$

Then  $\pi_{\xi_1}$  and  $\pi_{\omega\xi_2}$  represent the same parametrization of the leaf, so we get

$$\pi(\xi_1, z) = \pi(\omega\xi_2, z), \text{ for any } z \in \mathbb{C}.$$

In particular this gives  $\pi(\xi_1, z_1) = \pi(\omega\xi_2, z_1)$ . The projection  $\pi$  when restricted to  $\omega\xi_2 \times \mathbb{C} \rightarrow \mathcal{F}_{\xi_2}$  is injective. However, since  $\pi(\xi_1, z_1) = \pi(\xi_2, z_2)$  by hypothesis, we already know which point from  $\omega\xi_2 \times \mathbb{C}$  projects to  $\pi(\xi_1, z_1)$ . This is  $\gamma_{\frac{j}{2^k}}(\xi_2, z_2)$ , where

$\gamma_{\frac{j}{2^k}} \in \Gamma$  is the deck transform corresponding to the dyadic root of unity  $\omega = e^{2\pi i \frac{j}{2^k}}$ .

In conclusion  $\gamma_{\frac{j}{2^k}}(\xi_2, z_2)$  and  $(\omega\xi_2, z_1)$  must coincide. By Definition 9.10 it follows that  $\mathcal{O}_\Gamma(\xi_1, z_1) \sim_p \mathcal{O}_\Gamma(\xi_2, z_2)$ .  $\square$

**Remark 9.12.** It would be possible to identify  $\mathcal{M}_{p,a}/\sim_p$  with a quotient of  $J_p \times \mathbb{C}$  by an equivalence relation induced by the group orbits of  $\Gamma$  on  $\mathbb{S}^1 \times \mathbb{C}$ , using Remark 9.9.

Theorem 9.11 was proven in the context of perturbations of a hyperbolic polynomial with connected Julia set. The main ingredients were Theorems 4.2 and 2.3, out of which the first one is non-perturbative. The description of the critical locus from Theorem 2.3 may also hold throughout the entire hyperbolic component of the Hénon connectedness locus that contains perturbations of a hyperbolic quadratic polynomial with connected Julia set. The result of Theorem 9.11 could also be extended to this region.

## 10. EXTENSION TO SEMI-PARABOLIC HÉNON MAPS

Another extension concerns Hénon maps with a semi-parabolic fixed point, which come from perturbations of a polynomial with a parabolic fixed point.

**Definition 10.1.** A fixed point  $(x, y)$  of  $H$  is called semi-parabolic if the derivative  $DH_{(x,y)}$  has two eigenvalues  $|\mu| < 1$  and  $\lambda = e^{2\pi i p/q}$ .

The set of parameters  $(c, a) \in \mathbb{C}^2$  for which the Hénon map  $H_{c,a}$  has a fixed point with one eigenvalue a root of unity  $\lambda$ , is a curve of equation

$$\mathcal{P}_\lambda := \left\{ (c, a) \in \mathbb{C}^2 \mid c = (1+a) \left( \frac{\lambda}{2} + \frac{a}{2\lambda} \right) - \left( \frac{\lambda}{2} + \frac{a}{2\lambda} \right)^2 \right\}.$$

In [RT1] we studied Hénon maps with a semi-parabolic fixed point and small Jacobian (see also [RT2] for a discussion on a larger class of hyperbolic Hénon maps). Denote by  $V = \mathbb{D}_R \times \mathbb{D}_R$  the polydisk from the Hubbard filtration of  $\mathbb{C}^2$  depicted in Figure 1.

**Theorem 10.2** ([RT1]). *Let  $p(x) = x^2 + c_0$  be a polynomial with a parabolic fixed point of multiplier  $\lambda = e^{2\pi ip/q}$ . There exists  $a_0 > 0$  such that for all  $(c, a) \in \mathcal{P}_\lambda$  with  $0 \leq |a| < a_0$  the Hénon map  $H_{c,a}$  has connected Julia set  $J$  and there exists a homeomorphism  $\Phi : J_p \times \mathbb{D}_R \rightarrow J^+ \cap V$  such that the diagram*

$$\begin{array}{ccc} J_p \times \mathbb{D}_R & \xrightarrow{\Phi} & J^+ \cap V \\ \sigma \downarrow & & \downarrow H_{c,a} \\ J_p \times \mathbb{D}_R & \xrightarrow{\Phi} & J^+ \cap V \end{array}$$

*commutes. The function  $\sigma$  is given by  $\sigma(\xi, z) = \left(p(\xi), \xi + \frac{a}{2\xi}z\right)$ .*

It follows from Theorem 10.2 that for Hénon maps  $H_{c,a}$  which are small perturbations of the parabolic polynomial  $p$  inside the parabola  $\mathcal{P}_\lambda$ , the set  $J^+$  inside the polydisk  $V$  is a trivial fiber bundle over  $J_p$  with fibers biholomorphic to  $\mathbb{D}_R$ . Notice also that the vertical disks  $\zeta \times \mathbb{D}_R$ ,  $\zeta \in J_p$  that appear in the description of  $J^+ \cap V$  correspond to local stable manifolds of points from the Julia set  $J$ . The proof of Theorem 10.2 from [RT1] also implies that the foliation of  $U^+$  and the lamination of  $J^+$  fit together continuously. Therefore Theorem 4.2, which was known for hyperbolic maps, also holds true for this class of semi-parabolic Hénon maps.

The same arguments as in [LR] can be used to prove parts (a), (b), (d) and (e) of Theorem 2.3, when  $H_{c,a}$  is a small perturbation inside  $\mathcal{P}_\lambda$  of a quadratic polynomial with a parabolic fixed point. The reasoning is similar, because the critical point of a quadratic polynomial  $p$  with a parabolic or an attracting fixed point or cycle belongs to the interior of the filled-in Julia set  $K_p$ . So there exists a primary component of the critical locus  $\mathcal{C}_0$  inside  $U^+ \cap U^-$  asymptotic to the  $x$ -axis and there exists a biholomorphic extension of the function  $\varphi^+$  from  $\mathcal{C}_0 \cap V^+$  to  $\varphi^+ : \mathcal{C}_0 \rightarrow \mathbb{C} - \mathbb{D}$ . The polydisk  $V$  can be used as a trapping region for  $\mathcal{C}_0$  when  $a$  is small. Since  $J$  is connected from Theorem 10.2, the boundary  $\partial\mathcal{C}_0$  of the primary component is contained in  $J^+$ . The proof of part (c) from Theorem 2.3 follows from Theorem 10.2.

**Lemma 10.3.** *Let  $p$  be a quadratic polynomial with a parabolic fixed point of multiplier  $\lambda = e^{2\pi ip/q}$ . There exists  $a_0 > 0$  such that for all  $(c, a) \in \mathcal{P}_\lambda$  with  $0 \leq |a| < a_0$  the boundary  $\partial\mathcal{C}_0$  of the primary component  $\mathcal{C}_0$  of the critical locus for the Hénon map  $H_{c,a}$  is homeomorphic to the Julia set  $J_p$  of the parabolic polynomial  $p$ .*

Therefore Theorem 9.11 can be generalized to the semi-parabolic setting as follows

**Theorem 10.4.** *Let  $p$  be a quadratic polynomial with a parabolic fixed point of multiplier  $\lambda = e^{2\pi ip/q}$ . For all  $(c, a) \in \mathcal{P}_\lambda$  with  $0 < |a| < a_0$ , the group  $\Gamma_{c,a}$  acts properly discontinuously and without fixed points on  $\mathbb{S}^1 \times \mathbb{C}$ . Let  $\mathcal{M}_{c,a} = \mathbb{S}^1 \times \mathbb{C} / \Gamma_{c,a}$ . There exists a homeomorphism  $\hat{\pi}$  which makes the following diagram commute*

$$\begin{array}{ccc} \mathcal{M}_{c,a} / \sim_p & \xrightarrow{\hat{H}_{c,a}} & \mathcal{M}_{c,a} / \sim_p \\ \hat{\pi} \downarrow & & \downarrow \hat{\pi} \\ J^+ & \xrightarrow{H_{c,a}} & J^+ \end{array}$$



**Proof.** In view of Lemma 10.3 and Theorem 10.2, the proof is the same as that of Theorem 9.11.  $\square$

Notice also that the cocycle  $\alpha$  studied in Sections 5 and 6 is a full invariant of the family  $\mathcal{P}_\lambda$  when the Jacobian is small, because this family is parametrized by the eigenvalue  $\mu$  with  $|\mu| < 1$  of the semi-parabolic fixed point and  $\alpha(1)$  equals  $\mu$  by Condition 5.

## REFERENCES

- [BS1] E. Bedford, J. Smillie, *Polynomial diffeomorphisms of  $\mathbb{C}^2$ : currents, equilibrium measure and hyperbolicity*, Invent. Math. 103 (1991), no. 1, 69-99.
- [BS5] E. Bedford, J. Smillie, *Polynomial diffeomorphism of  $\mathbb{C}^2$ . V: Critical points and Lyapunov exponents*, J. Geom. Anal. 8(3) (1998), 349-383.
- [BS7] E. Bedford, J. Smillie, *Polynomial diffeomorphisms of  $\mathbb{C}^2$ . VII: Hyperbolicity and External Rays*, Ann. Sci. École Normale Sup. (4) 32 (1999), 455-497.
- [BS8] E. Bedford, J. Smillie, *Polynomial diffeomorphisms of  $\mathbb{C}^2$ . VIII: Quasi-Expansion*, American Journal of Mathematics 124 (2002), 221-271.
- [BV] G. T. Buzzard, K. Verma, *Hyperbolic Automorphisms and Holomorphic Motions in  $\mathbb{C}^2$* , Michigan Math. J. 49 (2001), 541-565.
- [DH] A. Douady, J. H. Hubbard, *Étude dynamique des polynômes complexes. Partie II*. Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], vol. 85, Université de Paris-Sud, Département de Mathématiques, Orsay, 1985.
- [FM] S. Friedland, J. Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems 9 (1989).
- [FS] J. E. Fornæss, N. Sibony, *Complex Hénon mappings in  $\mathbb{C}^2$  and Fatou-Bieberbach domains*, Duke Mathematics Journal, 65 (1992), 345-380.
- [F] T. Firsova, *Critical locus for Complex Hénon maps*, Indiana Math Journal, 61 (2012), 1603-1641; arXiv:1102.3924.
- [HOV1] J. H. Hubbard, R.W. Oberste-Vorth, *Hénon mappings in the complex domain I: The global topology of dynamical space*, Pub. Math. IHES 79 (1994), 5-46.
- [HOV2] J. H. Hubbard, R.W. Oberste-Vorth, *Hénon mappings in the complex domain II: Projective and inductive limits of polynomials*, in *Real and Complex Dynamical Systems*, Branner and Hjorth, eds., Kluwer Academic Publishers (1995), 89-132.
- [HOV3] J. H. Hubbard, R.W. Oberste-Vorth, *Linked solenoid mappings and the non-transversality locus invariant*, Indiana Univ. Math. J. 50 (2001), no. 1, 553-566.
- [K] A. Katok, *Combinatorial constructions in Ergodic Theory and Dynamics*, University Lecture Series, Volume 30 (2003).
- [LR] M. Lyubich, J. Robertson, *The Critical Locus and Rigidity of Foliations of Complex Hénon Maps*, manuscript, 2005.
- [M] J. Milnor, *Dynamics in one complex variable*, 3rd edn., Princeton Univ. Press, 2006.
- [MNTU] S. Morosawa, Y. Nishimura, M. Taniguchi, T. Ueda, *Holomorphic dynamics*, Cambridge Studies in Advanced Mathematics, 66. Cambridge University Press, Cambridge, 2000.
- [RT1] R. Radu, R. Tanase, *A structure theorem for semi-parabolic Hénon maps*, arXiv:1411.3824.
- [RT2] R. Radu, R. Tanase, *Semi-parabolic tools for hyperbolic Hénon maps and continuity of Julia sets in  $\mathbb{C}^2$* , Preprint, 2014.
- [T] R. Tanase, *Hénon maps, discrete groups, and continuity of Julia sets*, Ph.D. Thesis, Cornell University, 2013.
- [Th] W. Thurston, *On the geometry and dynamics of iterated rational maps*, in *Complex Dynamics: Families and Friends*, ed. by D. Schleicher, A.K. Peters, Wellesley/MA (2009), 3-137.